Particle creation in the effective action method

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Abstract

The effect of particle creation by nonstationary external fields is considered as a radiation effect in the expectation-value spacetime. The energy of created massless particles is calculated as the vacuum contribution in the energy-momentum tensor of the expectation value of the metric. The calculation is carried out for an arbitrary quantum field coupled to all external fields entering the general second-order equation. The result is obtained as a functional of the external fields. The paper gives a systematic derivation of this result on the basis of the nonlocal effective action. Although the derivation is quite involved and touches on many aspects of the theory, the result itself is remarkably simple. It brings the quantum problem of particle creation to the level of complexity of the classical radiation problem. For external fields like the electromagnetic or gravitational field there appears a quantity, the radiation moment, that governs both the classical radiation of waves and the quantum particle production. The vacuum radiation of an electrically charged source is considered as an example. The research is aimed at the problem of backreaction of the vacuum radiation.

Contents

1 Introduction

Introduction proper.

In the present paper we consider the problem of creation of particles from the vacuum by nonstationary external fields. This problem has been much discussed in the literature (see , e.g., [1-3]) but the present approach and the result obtained are new. Namely, we calculate the energy of massless particles created in external fields of arbitrary configuration and obtain the result as a functional of these fields. The calculation is done for a set of massless quantum fields coupled to all external fields entering the general second-order equation. The approach used is expectation-value theory and the method of the gravitational effective action.

The main result of the present work is briefly reported in [4]. This final result is simple but its derivation is not because the effect of particle creation is nonlocal and sits in the cubic terms of the effective action. This is an effect of the one-loop vertices. A use of the nonlocal effective action in the context of expectation values is almost unknown. Therefore, we present here a systematic derivation showing the techniques involved.

The result that obtains is remarkable since it brings the problem of the vacuum particle production to the level of complexity of the classical radiation problem. The strengths of external fields are expressed through their physical sources which are next integrated over certain spacelike hypersurfaces orthogonal to the geodesics. These integrals (we call them radiation moments) are direct generalizations of the moments of classical radiation theory. For external fields like the electromagnetic and gravitational fields, the radiation moments govern both the classical radiation of waves and the quantum particle production. As an example we consider the vacuum radiation produced by an electrically charged shell expanding in the self field.

The work is motivated by the fact that the problem with external fields is physically incomplete. For a restoration of the energy conservation law it should be regarded as a part of a dynamical problem for the field's expectation values in an initial quantum state. An outstanding example is the gravitational collapse problem [5]. In the expectation-

value problem one first calculates the currents as functionals of the mean fields, and next makes these fields subject to the self-consistent equations. At the stage of the calculation of the currents the expectation-value fields appear as external fields.

The setting of the problem with external fields in an asymptotically flat spacetime assumes that these fields become stationary in the remote past and future in which case there exist the standard in- and out- vacuum states for the quantum field [1]. Because the external field is nonstationary in the intermediate domain, the out-vacuum is generally a many-particle in-state, i.e. the external field creates particles from the in-vacuum. The method usually applied for the calculation of this effect consists in representing the quantum field as a sum over modes and obtaining the Bogoliubov transformation that relates the basis functions of the in- and out- modes.

The method of mode decomposition is normally used for an explicit solving of the equations in a given external field but one can also build a perturbation theory by solving for the basis functions iteratively in the external-field strength. This brings one to the loop diagrams of expectation-value theory, or the Schwinger-Keldysh diagrams [6-14]. As shown in [14], the Schwinger-Keldysh diagrams for the in-vacuum state are related through a certain set of rules to the Euclidean or Feynman effective action. This makes it possible to get away from both the mode decomposition and Schwinger-Keldysh diagrams, and carry out the calculation by the method of effective action.

The vacuum energy-momentum tensor is obtained by varying the loops of the effective action with respect to the metric (and next applying the rules of Ref.[14]). Therefore, irrespectively of the nature of the external field in question, one needs the effective action for the quantum field coupled to an external gravitational field (in addition to the external field in question). Since, for the effect of the vacuum patricle production, the lowest nonvanishing order is second order in the field strength, one needs the terms in the effective action that are quadratic in the field strength in question and linear in the gravitational field strength. The effect is thus contained in the one-loop triangular diagrams with at least one external gravitational line.

The covariant perturbation theory for the effective action is built in [14-18] where the one-loop vertices are calculated for all couplings of the quantum field whose small

disturbances are propagated by the general second-order equation. These results make the starting point for the present work.

Outline of the contents.

In the present section, after a brief outline of the contents of the paper, we introduce the field model for which the calculation will be carried out, and the limitations under which the result will be obtained. We introduce also some notions pertaining to various limits at the asymptotically flat infinity, and relate the vacuum particle production to the radiation in the expectation-value spacetime.

Sec.2 reviews the structure of the effective action and the procedure of obtaining from it the expectation-value current which in the present case is the energy-momentum tensor for the mean metric. The energy-momentum tensor contains a contribution of the in-vacuum, $T^{\mu\nu}_{\rm vac}$, which is the subject of study in the paper. For the radiation problem, $T^{\mu\nu}_{\rm vac}$ is needed only at the future null infinity ($\mathcal{I}^+$), and there are rules which facilitate obtaining $T^{\mu\nu}_{\rm vac}\Big|_{\mathcal{I}^+}$ greatly. These rules are also reviewed including a theorem which relates the limit of $\mathcal{I}^+$ for the kernel of an operator function to a certain limit for the function itself. This presentation is based on an earlier work [14-23] with the exception of the final result for $T^{\mu\nu}_{\rm vac}\Big|_{\mathcal{I}^+}$ which is obtained by a drastic simplification of the results in Ref.[18]. The technique of this simplification is outlined in Appendix A.

Sec.2 concludes with a discussion of the problem of averaging the quantum noise which is a sign-indefinite contribution in $T^{\mu\nu}_{\rm vac}\Big|_{\mathcal{I}^+}$ due to the quantum uncertainty. In two and four dimensions the mechanisms of this averaging are completely different since, in four dimensions, $T^{\mu\nu}_{\rm vac}\Big|_{\mathcal{I}^+}$ has the form of a total derivative in retarded time. The energy radiated by the vacuum for the whole history is thus determined by the limits of late time and early time, and, for this energy to be nonvanishing, certain nonlocal functions of external fields should exhibit a growth at late time. This offers a problem whose solution is the principal achievement of the present work.

The result for $T^{\mu\nu}_{\rm vac}\Big|_{\mathcal{I}^+}$ is obtained in Sec.2 in the form of a superposition of nonlocal operators acting on the sources of external fields. All nonlocal operators in $T^{\mu\nu}_{\rm vac}$ are defined

by their spectral forms in which the resolvent is the retarded Green function [14]. A derivation of the kernels of these operators is the subject of Sec.3. The resolvent is taken in the approximation in which it is determined by the geometrical two-point functions which, in their turn, are built on the basis of causal geodesics of the expectation-value (or background) spacetime. Therefore, all kernels in $T^{\mu\nu}_{\rm vac}$ inherit this geodetic structure and the retardation property. The geometrical objects involved are the light cone of a point, and the hyperboloid of equal (timelike) geodetic distance from a point. Subsequently, two other objects are derived from these: null and spacelike hyperplanes.

The behaviours of all kernels at $\mathcal{I}^+$ are obtained in Sec.4. In particular, it is shown that, at the limit of $\mathcal{I}^+$, the kernel of the operator $\log(-\Box)$ which stands for the external line in $T^{\mu\nu}_{\rm vac}\Big|_{\mathcal{I}^+}$ boils down to the kernel of $1/\Box$. Here, there first appear the radiation moments but only their ultrarelativistic limiting cases defined as integrals over the null hyperplanes. Upon the calculation of the behaviours of the vertex functions at $\mathcal{I}^+$, there emerges a problem. It turns out that, in the case of the external vector field (and only in this case), the superposition of kernels in $T^{\mu\nu}_{\rm vac}$ fails to converge at $\mathcal{I}^+$. The problem removes if the vector field is exposed to a special limitation that it contains no outgoing wave. The result in the paper is obtained under this limitation.

As mentioned above, the total energy of vacuum radiation is determined only by the limits of late time and early time. The purpose of Sec.5 is a proof that the limit of early time makes no contribution. The proof uses three facts: i) the retardation of all kernels in $T^{\mu\nu}_{\rm vac}$, ii) the presence of a time derivative in the kernel of $\log(-\Box)$, and iii) the assumed stationarity of the external fields in the past. Technically, it involves the properties of the geometric two-point functions and retarded kernels in presence of the Killing vector. The main result of this section is formulated as an assertion about the causality of the vacuum radiation. Sec.5 discusses also the question of convergence of the massless operators $1/\Box^n$ for $n > 1$. This question emerges in connection with some of the vertex operators in $T^{\mu\nu}_{\rm vac}$.

In Secs.6 and 7 we come to the heart of the matter: the behaviours at late time. Most of Sec.6 is devoted to the analysis of the behaviour of the function $\Big(1/\Box\Big)X$ in the future of $\mathcal{I}^+$. The result is that the dominant contribution to this behaviour comes from X at the limit i^+ which here is defined as the limit of infinite proper time along the timelike

geodesics that reach the future asymptotically flat infinity. It follows from this result that the radiation energy is given by an integral over the energies of the particles at i^+ and is nonvanishing only if the vertex functions have an appropriate *growth* at i^+.

The behaviours of the vertex functions at i^+ is the chief thing. They are obtained in Sec.7. Here, there appear the full radiation moments defined as integrals over the spacelike hyperplanes. Much work in Sec.7 is connected with the vector and tensor vertices. An alternative way of handling these vertices (which also requires much work) is considered in Appendix B.

The final result for the energy of vacuum radiation is presented in Sec.8, and it is gratifying to see its positivity. The positivity is based on two facts: i) the conservation of the currents of external fields, and ii) the self-adjointness of the equation of the quantum field. The discussion of the radiation moments is completed in Sec.8 by reviewing their role in classical radiation theory and establishing their relation to the textbook multipole moments. After that, it becomes visual that the quantum problem of particle creation is made almost the same thing as the classical problem of radiation of waves.

In Sec.9 the result for the vacuum radiation is specialized to spherical symmetry and to the external electromagnetic field. The longitudinal projections of the radiation moments are calculated. The vacuum radiation produced by an electrically charged spherical shell is considered as an example. The shell is assumed expanding in the self field from the state of maximum contraction to infinity, and the loss of its energy for the whole time of expansion is calculated in both the nonrelativistic and ultrarelativistic cases. Without accounting for the vacuum backreaction, the radiation of the ultrarelativistic shell violates the energy conservation law. The limits of validity of the technique based on the nonlocal expansion of the effective action are illustrated by specializing to the problem of particle creation in an external electric field.

The concluding remarks are devoted to a discussion of the limitations imposed on the external fields. In particular, the emergence of the limitation on the vector field signals that the theory contains another effect: the vacuum screening or amplification of the electromagnetic waves emitted by a source. This effect is equivalent to an observable renormalization of all multipole moments and is analogous to the effect of the vacuum

gravitational waves [23].

The model.

We shall consider the vacuum of a multicomponent quantum field $\varphi = \varphi^A$ coupled to external fields through the following equation for the small disturbances $\delta\varphi$:

$$\hat{H}\delta\varphi \equiv H^A_B \delta\varphi^B = 0 \tag{1.1}$$

with

$$\hat{H} = g^{\mu\nu}\nabla_\mu\nabla_\nu \hat{1} + \left(\hat{P} - \frac{1}{6}R\,\hat{1}\right) \quad . \tag{1.2}$$

Here the hat over a symbol indicates that this symbol is a matrix in the space of field components: $\hat{1} = \delta^A_{\ B}$, $\hat{P} = P^A_{\ B}$, etc. The matrix trace will be denoted tr. In (1.2), $\hat{P}$ is an arbitrary matrix potential, R is the Ricci scalar [1] of the metric $g_{\mu\nu}$, and ∇_μ is the covariant derivative with respect to any connection defining the commutator

$$[\nabla_\mu, \nabla_\nu]\delta\varphi = \hat{\mathcal{R}}_{\mu\nu}\delta\varphi \equiv \mathcal{R}^A_{\ B\mu\nu}\delta\varphi^B \ . \tag{1.3}$$

The full set of external fields is thus a metric, a connection, and a potential, and the respective set of field strengths is

$$\Re = \left(R^\alpha_{\ \beta\mu\nu}, \hat{\mathcal{R}}_{\mu\nu}, \hat{P}\right) \tag{1.4}$$

where $R^\alpha_{\ \beta\mu\nu}$ is the Riemann tensor of the metric in (1.2), and $\hat{\mathcal{R}}_{\mu\nu}$ is the commutator curvature in (1.3).

By solving the Jacobi and Bianchi identities

$$\nabla_\gamma\hat{\mathcal{R}}_{\alpha\beta} + \nabla_\beta\hat{\mathcal{R}}_{\gamma\alpha} + \nabla_\alpha\hat{\mathcal{R}}_{\beta\gamma} = 0 \ \ , \tag{1.5}$$

$$\nabla_\sigma R_{\mu\nu\beta}{}^\alpha + \nabla_\nu R_{\sigma\mu\beta}{}^\alpha + \nabla_\mu R_{\nu\sigma\beta}{}^\alpha = 0 \tag{1.6}$$

with the vacuum initial conditions [2], the commutator and Riemann curvatures can be expressed (nonlocally) through their contractions

$$\hat{J}^\mu \equiv \nabla_\nu\hat{\mathcal{R}}^{\mu\nu} \quad , \quad J^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R \tag{1.7}$$

[1] We use the conventions for which $[\nabla_\mu, \nabla_\nu]X^\alpha = R^\alpha_{\ \beta\mu\nu}X^\beta$, $R_{\alpha\beta} = R^\mu_{\ \alpha\mu\beta}$, $R = g^{\alpha\beta}R_{\alpha\beta}$.

[2] i.e. with the condition of absence of the incoming waves.

(see Refs. [15-20,23] and Appendix A below). The set of quantities

$$J = (J^{\mu\nu}, \hat{J}^{\mu}, \hat{P}) \tag{1.8}$$

represents the physical sources of external fields (1.4). Specifically, the commutator curvature $\hat{\mathcal{R}}_{\mu\nu}$ is a generalization of the Maxwell tensor, and $\hat{J}^{\mu}$ in (1.7) is a counterpart of the electromagnetic current. The expressions for the field strengths in terms of the currents are obtained iteratively but the conservation of the vector and tensor currents holds exactly:

$$\nabla_{\mu}\hat{J}^{\mu} = 0 \quad , \quad \nabla_{\mu}J^{\mu\nu} = 0 \quad . \tag{1.9}$$

Specifically, for the commutator curvature one has [24]

$$[\nabla_{\mu}, \nabla_{\nu}]\hat{\mathcal{R}}^{\alpha\beta} = \hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}^{\alpha\beta} - \hat{\mathcal{R}}^{\alpha\beta}\hat{\mathcal{R}}_{\mu\nu} + R^{\alpha}_{\ \gamma\mu\nu}\hat{\mathcal{R}}^{\gamma\beta} - R^{\beta}_{\ \gamma\mu\nu}\hat{\mathcal{R}}^{\gamma\alpha} \tag{1.10}$$

whence

$$\nabla_{\mu}\nabla_{\nu}\hat{\mathcal{R}}^{\mu\nu} = \frac{1}{2}\,[\nabla_{\mu}, \nabla_{\nu}]\hat{\mathcal{R}}^{\mu\nu} = 0 \ \ . \tag{1.11}$$

The vacuum energy of the quantum field φ is calculated below as a functional of the external-field strengths (1.4) to the lowest nonvanishing order in the number of loops and the power of $\Re$. Since solving the identities (1.5),(1.6) with the vacuum initial conditions is a part of the calculational procedure in covariant perturbation theory [15-18], the result will be expressed through the sources (1.8). At intermediate stages the potential term in (1.2)

$$\hat{Q} \equiv \hat{P} - \frac{1}{6}R\,\hat{1} \tag{1.12}$$

will as a whole figure in the capacity of a source but the final result will be expressed in terms of $\hat{P}$ for the reason explained in Sec.8.

The calculation in the paper is carried out under a number of limitations on the external fields whose significance is discussed in conclusion. One limitation is already predetermined: we consider only the fields of sources, i.e. neither the gravitational field nor the field represented by the commutator curvature will contain an incoming wave. It will be assumed that the sources of external fields have their supports in a spacetime tube with compact spatial sections and a timelike boundary. Their domain of nonstationarity will be assumed compact in both space and time. It will be assumed that the metric

has no singularities and horizons since here we consider only the case of finite particle production [3]. Finally, in the case of the commutator curvature there will be one futher limitation, namely that the vector source in (1.8) does not radiate classically. In the specific case where $\hat{\mathcal{R}}_{\mu\nu}$ is the Maxwell tensor, this limitation means that the external electromagnetic field contains no outgoing wave. No such limitation emerges in the case of the gravitational field.

The limits $\mathcal{I}^+$ and i^+ .

An important role in what follows is played by the limits along the null and timelike geodesics traced towards the future. By the assumption above, all causal geodesics reach the future null infinity or the future timelike infinity (see, e.g., [25]). Here we introduce the notations and reference equations pertaining to these limits that will next be used throughout the paper.

When dealing with the null geodesics, it is useful to build a Bondi-Sachs type [26,27] frame by choosing an arbitrary timelike geodesic (referred to as the central geodesic) and drawing the family of the future light cones with vertices on this geodesic (Fig.1). Let

$$u(x) = \text{const.} \quad , \quad (\nabla u)^2 \equiv 0 \tag{1.13}$$

(with ∇u past directed) be the equation of this family, $4\pi r^2(x)$ be the area of a 2 - dimensional section of a given cone in the induced metric, and $\phi(x)$ be a set of two coordinates labelling the null generators of a given cone. The $\phi(x)$ takes values on a 2-sphere and satisfies the orthogonality condition $(\nabla\phi, \nabla u) \equiv 0$. Then u is the retarded time, and r is the luminosity distance along the light rays that cross the central geodesic (the radial light rays).

One property of the Bondi-Sachs frame used in the paper is the fact that, if the point $\bar{x}$ is in the causal past of the point x, then $u(\bar{x}) \leq u(x)$. Indeed, let o be the point at which the central geodesic crosses the past light cone of x, and $\bar{x}$ be any point belonging

[3] i.e. finite energy production. The *number* of created massless particles may be infinite.

to this cone or its interior. The past light cone of x lies entirely outside the light cone of o as illustrated in Fig.2. We have $u(x) = u(o)$, and $u(\bar{x}) \leq u(o)$ since all points for which u is bigger than $u(o)$ are inside the future light cone of o.

When calculating only in the asymptotically flat domain of infinite luminosity distance, the compact domain may be cut out. The choice of the central geodesic is then immaterial, the retarded time is measured by an observer at infinity, and ϕ labels the points of the celestial 2-sphere $\mathcal{S}$ to which the radial light rays come as $r \to \infty$.

To every radial ray there corresponds a 2-parameter bundle of parallel rays that come simultaneously to the same point of the celestial sphere and are, therefore, indistinguishable at infinity. (They generate a null hyperplane discussed in Sec.4 .) Since the radial rays make a 3-parameter family, there is all together a 5-parameter family of light rays in spacetime but only three parameters are distinguishable in the principal approximation at infinity: u, ϕ. Changing the central geodesic signifies going over to the parallel rays.

We shall denote $\mathcal{I}^+[u, \phi, r \to \infty]$, or $\mathcal{I}^+[u, \phi]$, or $\mathcal{I}^+$, the limit of infinite luminosity distance r along the null geodesic that, when traced to the future, comes at the instant u of retarded time to the point ϕ of the celestial 2-sphere $\mathcal{S}$. By using the arbitrariness $u \to f(u)$, the retarded time will be normalized to coincide with the proper time of a distant observer at rest that registers the outgoing rays. This normalization condition reads

$$(\nabla u, \nabla r)\Big|_{\mathcal{I}^+} = -1 \quad , \tag{1.14}$$

and the asymptotic form of the metric in the Bondi-Sachs frame is

$$ds^2\Big|_{\mathcal{I}^+} = -du^2 - 2du dr + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1.15}$$

where $(\theta, \varphi) = \phi \in \mathcal{S}$. The integral over the 2-sphere $\mathcal{S}$ (normalized to have the area 4π) will be denoted

$$\int d^2\mathcal{S}(\phi) = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \quad . \tag{1.16}$$

To build a finite vector basis at $\mathcal{I}^+$ define the scalar

$$v\Big|_{\mathcal{I}^+} = 2r + u + O\Big(\frac{1}{r}\Big) \tag{1.17}$$

and the complex null vector tangent to the sphere $\mathcal{S}$

$$m_\alpha\Big|_{\mathcal{I}^+} = r(\nabla_\alpha\theta + \mathrm{i}\sin\theta\nabla_\alpha\varphi) + O(r^0) \quad . \tag{1.18}$$

The resultant null tetrad at $\mathcal{I}^+$

$$e_\alpha(\mu) = \nabla_\alpha u,\ \nabla_\alpha v,\ m_\alpha,\ m^*_\alpha \qquad (\mu = 1,2,3,4) \tag{1.19}$$

with μ labelling the vectors of the tetrad, and m^* complex conjugate to m satisfies the orthonormality relations

$$(\nabla u)^2 = (\nabla v)^2 = m^2 = (\nabla u,\ m) = (\nabla v,\ m) = 0 \quad ,$$

$$(\nabla u,\ \nabla v) = -2 \quad , \quad (m,\ m^*) = 2 \tag{1.20}$$

and can be used to expand the metric as follows:

$$g_{\mu\nu}\Big|_{\mathcal{I}^+} = -\frac{1}{2}(\nabla_\mu u\nabla_\nu v + \nabla_\mu v\nabla_\nu u) + \frac{1}{2}(m_\mu m^*_\nu + m^*_\mu m_\nu) \quad . \tag{1.21}$$

Consider now a timelike geodesic, and let s be the proper time along this geodesic. As $s \to \infty$, the particle moving along the geodesic will go out of the domain of nonstationarity of external fields with the energy E (per unit rest mass). Only the geodesics with $E > 1$ that reach the asymptotically flat infinity are relevant to the present discussion. We shall replace E with the *boost* parameter

$$\gamma = \frac{\sqrt{E^2-1}}{E} \quad , \quad 0 < \gamma < 1 \tag{1.22}$$

and denote $i^+[\gamma, \phi, s \to \infty]$, or $i^+[\gamma, \phi]$, or i^+, the limit $s \to \infty$ along the geodesic that comes to infinity with a given value of γ to a given point ϕ of the celestial 2-sphere.

A distinction of this case from the case of null geodesics (apart from $\gamma = 1$ in the latter case) is that the timelike geodesics differing by translations including the time translations are indistinguishable at infinity. There are again three parameters that register at infinity but the parameters are now γ and ϕ, and there is a 3-parameter congruence of the geodesics that come to infinity with one and the same values of γ and ϕ. (This congruence will be considered in Sec.7 .) All together, there is a 6-parameter family of timelike geodesics that come to infinity.

The Bondi-Sachs frame can be used also at i^+ with the asymptotic form of the metric in (1.15). For the asymptotically flat metric at $i^+(\gamma > 0)$ one may use the vector basis (1.21) or the related basis

$$g_{\mu\nu}\Big|_{i^+} = -\nabla_\mu t \nabla_\nu t + \nabla_\mu r \nabla_\nu r + \frac{1}{2}(m_\mu m_\nu^* + m_\mu^* m_\nu) \tag{1.23}$$

with

$$t\Big|_{i^+} = r + u + \text{const.} \quad , \quad \nabla u = \nabla t - \nabla r \quad , \quad \nabla v = \nabla t + \nabla r \quad . \tag{1.24}$$

The parameters ϕ of a point at i^+ or $\mathcal{I}^+$ can be replaced by a unit direction vector at infinity whose spatial components in the Minkowski frame at i^+

$$ds^2\Big|_{i^+} = -\, dt^2 + \delta_{ik} d\mathbf{x}^i d\mathbf{x}^k \tag{1.25}$$

will be denoted $n_i = n_i(\phi), \quad i = 1, 2, 3$. With the Euler parametrization $\phi = (\theta, \varphi)$ in (1.15),

$$n_1(\phi) = \sin\theta \sin\varphi \quad , \quad n_2(\phi) = \sin\theta\cos\varphi \quad , \quad n_3(\phi) = \cos\theta \quad , \tag{1.26}$$

and

$$\mathbf{x}^i = r\, n^i(\phi) \quad , \quad n^i = \delta^{ik} n_k \quad , \quad n^i n_k = 1 \quad . \tag{1.27}$$

The basis vectors in (1.23) have in the Minkowski frame (1.25) the components

$$\nabla_\mu r = \delta_\mu{}^i n_i \quad , \quad \frac{1}{2}(m_\mu m_\nu^* + m_\mu^* m_\nu) = \delta_\mu{}^i \delta_\nu{}^k (\delta_{ik} - n_i n_k) \quad . \tag{1.28}$$

The limits i^+ and $\mathcal{I}^+$ are related. As $\gamma \to 1$ the geodesic at i^+ approaches the null geodesic that comes to $\mathcal{I}^+$ at late time. Therefore, for an analytic function X, the sequence of limits i^+ and $\gamma \to 1$ should coincide with the future of $\mathcal{I}^+$:

$$\Big(X\Big|_{i^+[\gamma,\phi,s\to\infty]}\Big)_{\gamma\to 1} = \Big(X\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]}\Big)_{u\to\infty} \tag{1.29}$$

under the substitutions

$$s = \frac{\sqrt{1-\gamma^2}}{\gamma}\, r \quad , \quad \gamma = 1 - \frac{u}{r} + O\Big(\frac{u^2}{r^2}\Big) \quad . \tag{1.30}$$

Vacuum radiation in the expectation-value spacetime.

In the framework of the problem of particle creation by an external field, it is impossible to answer the question where the energy of the created real particles comes from. Clearly, the vacuum particle production is only a mechanism of the energy transfer. The energy comes ultimately from the external field but this answer assumes that the external field should stop being external.

In the self-consistent setting of the problem, the external fields become expectation values evolving from an initial quantum state. Specifically in the presently considered model, φ^A is the full set of quantum fields. Some of these fields have nonvanishing expectation values represented by the set of curvatures in (1.4). The expectation-value equations are obtained by the rules of Ref. [14] from the action

$$\frac{1}{16\pi}\int dx g^{1/2} R + S_{\rm source} + S_{\rm vac} \tag{1.31}$$

where $S_{\rm source}$ is the (renormalized) classical action for $\hat{\mathcal{R}}_{\mu\nu}$ and $\hat{P}$, and $S_{\rm vac}$ is the sum of all vacuum loops. The equations for the expectation value of the metric can be written in the form of the Einstein equations

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi \Big(T^{\mu\nu}_{\rm source} + T^{\mu\nu}_{\rm vac} \Big) \quad , \tag{1.32}$$

$$T^{\mu\nu}_{\rm source} = \frac{2}{g^{1/2}} \frac{\delta S_{\rm source}}{\delta g_{\mu\nu}} \tag{1.33}$$

in which there appears some extra source $T^{\mu\nu}_{\rm vac}$, the energy-momentum tensor of the in-vacuum. This source is a subject of the calculation below. Eq. (1.32) should be supplemented with similar equations for $\hat{\mathcal{R}}_{\mu\nu}$ and $\hat{P}$ (and, of course, there should be equations for the higher-order correlation functions, also derivable from (1.31), which will not be discussed here).

The energy conservation law in the expectation-value spacetime is a consequence of Eq. (1.32) with the asymptotically flat boundary conditions. It is expressed in the existence of a conserved ADM mass $M_{\rm ADM}$ which equals the total energy of all sources and waves in the initial state, and a non-conserved Bondi mass $M(u)$ defined by the metric at $\mathcal{I}^+$

and accounting for radiation [25]. The difference $M(-\infty) - M(u)$ is the energy radiated from an isolated system by the instant u of retarded time, and

$$M(-\infty) = M_{\mathrm{ADM}} \quad . \tag{1.34}$$

The equation of the energy balance, or the Bondi-Sachs equation [26-27,23], is of the form

$$\begin{aligned} -\frac{d\mathrm{M}(u)}{du} &= \frac{1}{4\pi}\int d^2\mathcal{S}(\phi)\left|\frac{\partial}{\partial u}\,\mathbf{C}_{\mathrm{grav}}(u,\phi)\right|^2 \\ &+ \int d^2\mathcal{S}(\phi)\,\Big(\frac{1}{4}\,r^2\,T^{\mu\nu}_{\mathrm{source}}\nabla_\mu v\nabla_\nu v + \frac{1}{4}\,r^2\,T^{\mu\nu}_{\mathrm{vac}}\nabla_\mu v\nabla_\nu v\Big)\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} \end{aligned} \tag{1.35}$$

and is to be solved with the initial condition (1.34). The first term in (1.35) is the energy flux of the outgoing gravitational waves ($\partial\mathbf{C}_{\mathrm{grav}}/\partial u$ is the complex gravitational news function [26,27] defined by the metric at $\mathcal{I}^+$), and the remaining terms are the energy fluxes through $\mathcal{I}^+$ of all sources of the *mean* gravitational field including the vacuum. The news function also contains a contribution induced by the vacuum [23].

We are presently interested only in the contribution of $T^{\mu\nu}_{\mathrm{vac}}$ to the radiation flux. This contribution will be nonvanishing only when the fields solving the expectation-value equations create real massless particles from the vacuum. If the other contributions are absent [4] :

$$-\frac{d\mathrm{M}(u)}{du} = \int d^2\mathcal{S}(\phi)\,\Big(\frac{1}{4}r^2 T^{\mu\nu}_{\mathrm{vac}}\nabla_\mu v\nabla_\nu v\Big)\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} \tag{1.36}$$

then the total radiated energy equals the total energy of created particles:

$$\mathrm{M}(-\infty) - \mathrm{M}(\infty) = \int\limits_{-\infty}^{\infty}\Big(-\frac{d\mathrm{M}}{du}\Big)du = \sum_p \epsilon_p < \mathrm{in\ vac}\,|a^{+p}_{\mathrm{out}}a^{p}_{\mathrm{out}}|\,\mathrm{in\ vac} > \tag{1.37}$$

(see, e.g., [22]). Here $a^{+p}_{\mathrm{out}}, a^{p}_{\mathrm{out}}$ are the creation and annihilation operators for the quanta of the field φ in the out-vacuum, and ϵ_p is the energy in the out-mode p.

The inference from Eq. (1.37) is that the massless particles created from the vacuum are radiated through the future null infinity of the expectation-value spacetime. As seen from the initial condition (1.34), this vacuum radiation takes its energy from the ADM

[4] As is the case, for example, if the radiation of waves, both gravitational and matter, is banned by the symmetry of the initial state.

mass of the expectation value of the metric. Since the ADM mass is conserved, it can be calculated on a spacelike hypersurface taken in the remote past. The $T^{\mu\nu}_{\rm vac}$ is a retarded functional of the mean fields, and it vanishes in the remote past (see Sec.5 below). Therefore, the ADM mass remains unaffected by quantum corrections and equal to the energy in $T^{\mu\nu}_{\rm source}$ calculated on an initial hypersurface (plus the energy of an incoming gravitational wave if any) [5]. It thus equals the energy of the "external" fields in the past from their domain of nonstationarity. In this way the energy conservation is restored.

Eq. (1.36) with $T^{\mu\nu}_{\rm vac}$ obtained from the effective action will be used below to calculate the quantity (1.37). Special attention will be payed to the positivity of this quantity.

[5]This has an important consequence that the ADM mass of the mean metric remains positive if $T^{\mu\nu}_{\rm source}$ is energy-dominant [16].

2 The vacuum energy-momentum tensor

The effective action.

The one-loop vacuum action for the field φ is

$$S_{\rm vac} = \frac{\rm i}{2}{\rm Tr}\log\hat{H} \tag{2.1}$$

where Tr denotes the functional trace, and $\hat{H}$ is the operator (1.2). Upon the calculation of the loop [14-18], this action takes the form

$$\begin{aligned} S_{\rm vac} &= S(2) + S(3) + O[\Re^4] \quad , && (2.2)\\ S(2) &= \frac{1}{2(4\pi)^2}\int dx g^{1/2}\,{\rm tr}\sum_{i=1}^{5}\gamma_i(-\Box_2)\Re_1\Re_2(i) \quad , && (2.3)\\ S(3) &= \frac{1}{2(4\pi)^2}\int dx g^{1/2}\,{\rm tr}\sum_{i=1}^{29}\Gamma_i(-\Box_1,-\Box_2,-\Box_3)\Re_1\Re_2\Re_3(i) \quad , && (2.4)\\ \Box &\equiv g^{\mu\nu}\nabla_\mu\nabla_\nu && (2.5)\end{aligned}$$

which is the general form of a functional of the field strengths (1.4) expanded over a basis of nonlocal invariants [20]. The term $S(2)$ is a linear combination of five basis invariants $\Re_1\Re_2(i)$ of second order in $\Re$ with the operator coefficients (form factors) $\gamma_i(-\Box)$. The term $S(3)$ is a linear combination of twenty nine basis invariants $\Re_1\Re_2\Re_3(i)$ of third order in $\Re$ with the form factors $\Gamma_i(-\Box_1,-\Box_2,-\Box_3)$. A calculation of the effective action in any specific model or approximation boils down to obtaining the form factors in (2.2). The expansion (2.2) can also be assumed as a basis of a phenomenological theory of the vacuum [19-22].

The basis invariants of second order in $\Re$ and their respective one-loop form factors are of the form [15]

$$\begin{aligned} \Re_1\Re_2(1) &= R_{1\mu\nu}R_2^{\mu\nu}\hat{1} \quad , && (2.6)\\ \Re_1\Re_2(2) &= R_1R_2\hat{1} \quad , && (2.7)\\ \Re_1\Re_2(3) &= \hat{P}_1R_2 \quad , && (2.8)\\ \Re_1\Re_2(4) &= \hat{P}_1\hat{P}_2 \quad , && (2.9)\\ \Re_1\Re_2(5) &= \hat{\mathcal{R}}_{1\mu\nu}\hat{\mathcal{R}}_2^{\mu\nu} \quad , && (2.10)\end{aligned}$$

$$\gamma_1(-\Box) = \frac{1}{60}\Big(-\log\Big(-\frac{\Box}{c^2}\Big) + \frac{16}{15}\Big) \quad , \tag{2.11}$$

$$\gamma_2(-\Box) = \frac{1}{180}\Big(\log\Big(-\frac{\Box}{c^2}\Big) - \frac{37}{30}\Big) \quad , \tag{2.12}$$

$$\gamma_3(-\Box) = -\frac{1}{18} \quad , \tag{2.13}$$

$$\gamma_4(-\Box) = -\frac{1}{2}\log\Big(-\frac{\Box}{c^2}\Big) \quad , \tag{2.14}$$

$$\gamma_5(-\Box) = \frac{1}{12}\Big(-\log\Big(-\frac{\Box}{c^2}\Big) + \frac{2}{3}\Big) \tag{2.15}$$

where the parameter $c^2 > 0$ accumulates the ultraviolet arbitrariness. This parameter doesn't affect the vacuum current at null infinity (see [21] and the calculation below).

The third-order action contains no arbitrary parameters. Among the basis invariants of third order in $\Re$, eleven contain no derivatives, for example

$$\Re_1\Re_2\Re_3(1) = \hat{P}_1\hat{P}_2\hat{P}_3 \quad , \tag{2.16}$$

fourteen contain two derivatives, for example

$$\Re_1\Re_2\Re_3(12) = \hat{\mathcal{R}}_1^{\alpha\beta}\nabla^\mu\hat{\mathcal{R}}_{2\mu\alpha}\nabla^\nu\hat{\mathcal{R}}_{3\nu\beta} \quad , \tag{2.17}$$

three contain four derivatives, for example

$$\Re_1\Re_2\Re_3(26) = \nabla_\alpha\nabla_\beta R_1^{\mu\nu}\nabla_\mu\nabla_\nu R_2^{\alpha\beta}\hat{P}_3 \quad , \tag{2.18}$$

and one contains six derivatives :

$$\Re_1\Re_2\Re_3(29) = \nabla_\lambda\nabla_\sigma R_1^{\alpha\beta}\nabla_\alpha\nabla_\beta R_2^{\mu\nu}\nabla_\mu\nabla_\nu R_3^{\lambda\sigma}\hat{1} \quad . \tag{2.19}$$

The full table of third-order invariants and their one-loop form factors is given in [17].

Since the operator arguments of the form factors in (2.4) commute, the form factors themselves are ordinary functions [6] . As analytic functions, they are defined by their spectral forms (in each argument) with the resolvents $1/(\Box - m^2)$.

The action in the form (2.2)-(2.4) (i.e. with the loop done) determines both the matrix elements between the in- and out- vacua and the expectation values in the in-vacuum [14]. The difference is in the boundary conditions for the resolvents of the nonlocal operators.

[6] Beyond third order in $\Re$ this is no more the case [19,20].

When the action (2.2) is varied, the resolvents are regarded as obeying the variational rule

$$\delta\,\frac{1}{\Box - m^2} = -\frac{1}{\Box - m^2}\,\delta\Box\,\frac{1}{\Box - m^2}\quad , \tag{2.20}$$

and, after the variation has been completed, they are identified with the Feynman Green functions in the case of matrix elements and with the retarded Green functions in the case of expectation values [14]. Introducing a notation for the result of this procedure, we may write for $T^{\mu\nu}_{\rm vac}$ in (1.32) the expression

$$T^{\mu\nu}_{\rm vac} = \frac{2}{g^{1/2}}\frac{\delta S_{\rm vac}}{\delta g_{\mu\nu}}\bigg|_{\Box\to\Box_{\rm ret}}\quad . \tag{2.21}$$

For obtaining $T^{\mu\nu}_{\rm vac}$ from the action (2.2) one needs to know the variational derivatives of the commutator curvature and potential with respect to the metric. The dependence of $\hat{\mathcal{R}}_{\mu\nu}$ and $\hat{P}$ on the metric is different in different models [24]. The calculation below is carried out for the case where $\hat{\mathcal{R}}_{\mu\nu}$ and $\hat{P}$ are metric independent but, as explained in conclusion, the final result is valid for arbitrary local $\hat{\mathcal{R}}_{\mu\nu}$ and $\hat{P}$.

Obtaining $T^{\mu\nu}_{\rm vac}$ at $\mathcal{I}^+$.

The $T^{\mu\nu}_{\rm vac}(x)$ is here to be calculated only for x at $\mathcal{I}^+$, and only the terms that contribute to the energy flux through $\mathcal{I}^+$ are to be retained. These are the terms of order $r^{-2}(x)$, $x\to\mathcal{I}^+$ with r the luminosity distance. Below, the notation $T^{\mu\nu}_{\rm vac}\big|_{\mathcal{I}^+}$ is used for $T^{\mu\nu}_{\rm vac}$ calculated up to terms $O(1/r^3)$, and the terms $O(1/r^3)$ are referred to as vanishing at $\mathcal{I}^+$.

When computing $T^{\mu\nu}_{\rm vac}\big|_{\mathcal{I}^+}$ from the action (2.2) it is useful to have at hand the following behaviours that we quote from Refs.[21-23] and Sec.4 below:

$$\frac{1}{\Box}X\bigg|_{\mathcal{I}^+} = O\Big(\frac{1}{r}\Big)\quad ,\quad \log(-\Box)X\bigg|_{\mathcal{I}^+} = O\Big(\frac{1}{r^2}\Big)\quad , \tag{2.22}$$

$$\frac{\log(\Box_1/\Box_2)}{\Box_1 - \Box_2}X_1\,X_2\bigg|_{\mathcal{I}^+} = O\Big(\frac{1}{r^3}\Big)\quad . \tag{2.23}$$

The behaviours of these and other form factors including the higher-order ones can be obtained from the behaviour of the resolvent operator [21,22]:

$$\frac{1}{\Box - m^2}\,X\bigg|_{\mathcal{I}^+} \propto \frac{1}{r}\,\exp(-|{\rm const.}|m\sqrt{r})\quad . \tag{2.24}$$

The behaviours of the field strengths (1.4) and their sources (1.8) are in the most general case (see, e.g., Sec.8 below)

$$\Re\Big|_{\mathcal{I}^+} = O\Big(\frac{1}{r}\Big) \quad , \quad J\Big|_{\mathcal{I}^+} = O\Big(\frac{1}{r^2}\Big) \tag{2.25}$$

and, with the present assumptions, these behaviours soften. Hence it is seen that the terms of $T^{\mu\nu}_{\rm vac}(x)$ in which the curvature $\Re$ appears at the observation point x can often be discarded at $\mathcal{I}^+$. Also the terms in which there forms a positive integer power of the operator $\Box$ acting at the observation point can often be discarded since

$$\Box X\Big|_{\mathcal{I}^+} = O\left(\frac{1}{r}\,X\right) \quad . \tag{2.26}$$

Another useful fact is

$$g^{\mu\nu}\nabla_\mu X_1 \nabla_\nu X_2\Big|_{\mathcal{I}^+} = O\Big(\frac{1}{r}X_1X_2\Big) \quad . \tag{2.27}$$

Both (2.26) and (2.27) follow from the form of the metric at $\mathcal{I}^+$, Eq. (1.15).

Using Eq. (2.23) it has been shown in Ref. [23] that the variations of second-order form factors γ make a vanishing contribution at $\mathcal{I}^+$. This fact makes an essential difference with the case of two dimensions (see below). Owing to this fact, the only contribution of the action $S(2)$ to $T^{\mu\nu}_{\rm vac}$ that survives at $\mathcal{I}^+$ comes from varying the curvatures $\Re$ in the products $\Re_1\Re_2(i)$.

Since varying destroys the curvature, the effective action of third order in $\Re$ enables one to obtain $T^{\mu\nu}_{\rm vac}$ only to second order in $\Re$. Therefore, in the action $S(3)$ only the curvatures $\Re$ in the products $\Re_1\Re_2\Re_3(i)$ need to be varied. The contributions of variations of the third-order form factors Γ are already $O[\Re^3]$. It is also useful to keep in mind that any commutator of the derivatives ∇ with each other or with the form factors is $O[\Re]$.

Using the rules above to discard the terms vanishing at $\mathcal{I}^+$ we obtain

$$\begin{aligned} T^{\mu\nu}_{\rm vac}\Big|_{\mathcal{I}^+} &= \frac{1}{(4\pi)^2}\,{\rm tr}\,\nabla^\mu\nabla^\nu\Big(\gamma_1(-\Box) + 2\gamma_2(-\Box)\Big)R\hat{1} \\ &+ \frac{1}{(4\pi)^2}\,{\rm tr}\,\Big\{\nabla^\mu[\nabla_\alpha,\gamma_1(-\Box)]R^{\alpha\nu} + \nabla^\nu[\nabla_\alpha,\gamma_1(-\Box)]R^{\alpha\mu} \\ &\quad -\frac{1}{2}\nabla^\mu[\nabla^\nu,\gamma_1(-\Box)]R - \frac{1}{2}\nabla^\nu[\nabla^\mu,\gamma_1(-\Box)]R\Big\}\hat{1} \\ &+ \frac{2}{g^{1/2}}\,\frac{\delta S(3)}{\delta g_{\mu\nu}} + O[\Re^3] \end{aligned} \tag{2.28}$$

where the variation of the action $S(3)$ is not yet calculated but there appear the commutators which are of the same order $\Re^2$ as the variation of $S(3)$. The commutators emerge when the terms linear in $\Re$ are brought to their form in (2.28) with the aid of the Bianchi identities. Taking these commutators into account is necessary for $T^{\mu\nu}_{\rm vac}$ to have the correct form at $\mathcal{I}^+$ (Eq. (2.36) below). Both commuting and varying of the operator functions is accomplished with the aid of their spectral forms [17,23]. By (2.21), all nonlocal operators in (2.28) have the retarded boundary conditions.

Consider now the variation of the action $S(3)$. A generic term in $\delta S(3)$ is of the form

$$\int dx g^{1/2}\Gamma(-\Box_1,-\Box_2,-\Box_3)\delta\Re_1\,\Re_2\,\Re_3+O[\Re^3] \tag{2.29}$$

where $\delta\Re$ has the structure

$$\delta\Re(x)=\nabla\nabla\delta g(x)+O[\Re] \quad . \tag{2.30}$$

Since, when going over to the variational derivative, the operator acting on $\delta g(x)$ transposes, the respective term in the variational derivative at a point x has the following structure:

$$\nabla\nabla\Gamma(-\Box,-\Box_2,-\Box_3)\Re(x_2)\Re(x_3)\Big|_{x_2=x_3=x} \tag{2.31}$$

where the operators $\nabla\nabla$ and the first argument $-\Box$ of the form factor act at the observation point x. In (2.31), first $\Box_2$ acts on $\Re(x_2)$ and $\Box_3$ acts on $\Re(x_3)$ with subsequently making the points x_2 and x_3 coincident with the observation point x, and next the argument $-\Box$ of the form factor and the derivatives $\nabla\nabla$ act on the thus obtained function of the observation point.

The important fact is that one of the operator arguments of Γ, the one that in (2.29) acts on $\delta\Re$, becomes an overall operator in (2.31). Therefore, the theorem in [21,22] applies by which the behaviour of the function (2.31) as $x\to\mathcal{I}^+$ is determined by the behaviour of the function $\Gamma(-\Box,-\Box_2,-\Box_3)$ as $-\Box\to 0$ with $-\Box$ the argument that becomes an overall operator in (2.31). For the function (2.31) to be $O(1/r^2)$, the form factor should be [21]

$$\Gamma(-\Box,-\Box_2,-\Box_3)=O\Big(\log(-\Box)\Big) \quad , \quad -\Box\to 0 \quad . \tag{2.32}$$

The $\log(-\Box)$ *terms of the form factors at small* $\Box$ *determine the* $1/r^2$ *terms of* $T^{\mu\nu}_{\rm vac}$ *at* $\mathcal{I}^+$ [21]. Eq. (2.22) anticipates this fact.

Thus, for the calculation of $T^{\mu\nu}_{\rm vac}$ at $\mathcal{I}^+$, one doesn't need the exact form factors. One needs only their asymptotic behaviours in each argument at the limit where this argument is small, and the remaining arguments are fixed. These asymptotic behaviours are presented in [18]. All the form factors Γ behave as follows ($k = 1, 2, 3$):

$$\begin{aligned}\Gamma_i(-\Box_1, -\Box_2, -\Box_3) \;&=\; \frac{1}{\Box_k} A_i^{\,k}(\Box_m, \Box_n) + \log(-\Box_k) B_i^{\,k}(\Box_m, \Box_n) + O(\Box_k^0) \;, \\ & -\Box_k \to 0 \;\;,\;\; m \neq k, \;\; n \neq k, \;\; m < n \end{aligned} \tag{2.33}$$

where $A_i^{\,k}$ and $B_i^{\,k}$ are functions of the two arguments $\Box$ other than $\Box_k$. These behaviours differ from (2.32) since, in addition to the expected $\log(-\Box)$ terms, they contain senior $1/\Box$ terms. However, as shown in [23], the $1/\Box$ terms exactly cancel in the energy-momentum tensor. The variational derivative of $S(3)$ is a sum of contributions of the form (2.31), and the form factors Γ enter this sum only in certain linear combinations. The $1/\Box$ terms cancel in these combinations leaving the $\log(-\Box)$ terms as the leading asymptotic terms at $\Box \to 0$ [23].

Thus we infer that the contributions of the form factors Γ to $T^{\mu\nu}_{\rm vac}\big|_{\mathcal{I}^+}$ are completely determined by the coefficient $B_i^{\,k}$ in (2.33). The table of these coefficients is given in [18]. The simplest ones are the coefficients in the form factor Γ_1 of the basis invariant (2.16). They are of the form

$$B_1^{\,k}(\Box_m, \Box_n) = \frac{1}{3}\,\frac{\log(\Box_m/\Box_n)}{\Box_m - \Box_n} \tag{2.34}$$

for all the three values of k. The remaining $B_i^{\,k}$ are obtained by differentiating the generating expression

$$\frac{\log(j_m\Box_m/j_n\Box_n)}{j_m\Box_m - j_n\Box_n} \tag{2.35}$$

with respect to the auxiliary variables j_m, j_n and subsequently setting $j_m = j_n = 1$ [17,18]. All $B_i^{\,k}$ are the thus obtained functions having also rational coefficients of the form $1/\Box_n$ or $\Box_m/\Box_n$ and rational additions.

The table of the functions $B_i^{\,k}$ in [18] can be considerably simplified with the aid of the technique outlined in Appendix A. The use of this technique makes it possible to bring $T^{\mu\nu}_{\rm vac}\big|_{\mathcal{I}^+}$ to the final form below.

The result for $T^{\mu\nu}_{\rm vac}$ at $\mathcal{I}^+$.

The result for $T^{\mu\nu}_{\rm vac}\Big|_{\mathcal{I}^+}$ is of the form

$$T^{\mu\nu}_{\rm vac}\Big|_{\mathcal{I}^+} = \frac{1}{(4\pi)^2}\ \nabla^\mu\nabla^\nu \log(-\Box)\ I(x) \tag{2.36}$$

and is determined by a single scalar $I(x)$:

$$I(x) = {\rm tr}\hat{I}_1(x) + {\rm tr}\hat{I}_2(x) + O[\Re^3] \tag{2.37}$$

where $\hat{I}_1(x)$ and $\hat{I}_2(x)$ are the contributions of first and second order in $\Re$. The contribution of first order obtains directly from (2.28) and from the expressions (2.11)-(2.12) for the second-order form factors:

$$\hat{I}_1 = -\frac{1}{180}\ R\hat{1} \quad . \tag{2.38}$$

The contribution of second order obtains from the results in [18] with the aid of the technique of Appendix A. It is of the following form:

$$\hat{I}_2 = \nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu\ \hat{V}^{\alpha\beta\mu\nu}_{\rm grav} + \nabla_\alpha\nabla_\beta\hat{V}^{\alpha\beta}_{\rm vect} + \nabla_\alpha\nabla_\beta\hat{V}^{\alpha\beta}_{\rm cross} + \hat{V}_{\rm scalar} + {\rm TREES} \quad , \tag{2.39}$$

$$\hat{V}^{\alpha\beta\mu\nu}_{\rm grav} = \frac{1}{\Box_1\Box_2}\Big(\frac{1}{180}F(3,3) - \frac{1}{36}F(2,2)\Big)J_1{}^{\alpha\beta}J_2{}^{\mu\nu}\hat{1} \quad , \tag{2.40}$$

$$\hat{V}^{\alpha\beta}_{\rm vect} = -\frac{1}{\Box_1\Box_2}\Big(\frac{1}{6}F(2,2) - \frac{1}{3}F(1,1)\Big)\hat{J}_1{}^\alpha\hat{J}_2{}^\beta \quad , \tag{2.41}$$

$$\hat{V}^{\alpha\beta}_{\rm cross} = -\frac{1}{\Box_2}\Big(\frac{1}{6}F(2,2) - \frac{1}{3}F(1,1)\Big)J_1{}^{\alpha\beta}\hat{Q}_2 \quad , \tag{2.42}$$

$$\hat{V}_{\rm scalar} = \Big(\frac{1}{2}F(1,1) - \frac{1}{6}F(0,0)\Big)\hat{Q}_1\hat{Q}_2 \tag{2.43}$$

where $J^{\alpha\beta}$, $\hat{J}^\alpha$, $\hat{Q}$ are the sources of external fields in (1.8) and (1.12), and $F(n,n)$ with $n = 0,1,2,3$ are specific cases of the *vertex* operator

$$F(m,n) = \Big(\frac{\partial}{\partial j_1}\Big)^m\Big(\frac{\partial}{\partial j_2}\Big)^n \frac{\log(j_1\Box_1/j_2\Box_2)}{j_1\Box_1 - j_2\Box_2}\Big|_{j_1=j_2=1} \quad . \tag{2.44}$$

The V 's in (2.39) will be referred to as the vertex terms. As distinct from the V 's, the trees [7] in (2.39) are the terms whose operator coefficients factorize into functions of one

[7]The name "tree" is used here not quite in the usual sense. It should also be clear that both the vertices and trees are terms of *the calculated loop*.

variable:

$$\text{TREES} = \frac{1}{90}\Big(\nabla_\beta \frac{1}{\Box} J^{\alpha\lambda}\Big)\Big(\nabla_\alpha \frac{1}{\Box} J^\beta_\lambda\Big)\hat{1} + \frac{1}{180}\; J^{\alpha\beta}\Big(\frac{1}{\Box} J_{\alpha\beta}\Big)\hat{1} - \frac{1}{360} R\Big(\frac{1}{\Box} R\Big)\hat{1} \quad . \tag{2.45}$$

The fact that only the sources of external fields appear in $T^{\mu\nu}_{\rm vac}$ is a result of the use of the Jacobi and Bianchi identities in the calculation of the nonlocal effective action [15-20]. As will be seen, the conservation laws (1.9) for the vector and tensor sources play a crucial role in the consistency of the result above.

The present calculation reveals a number of nontrivial properties of the effective action such as Eq. (2.36). When working with the explicit one-loop form factors these properties emerge as a result of mysterious cancellations. In fact they can be predicted on the basis of the axiomatic approach to the effective action [19-22]. In particular, the behaviours (2.33) follow from the requirement of asymptotic flatness of the expectation-value spacetime and should hold for all form factors to all loop orders. The fact that the $1/\Box$ terms of these behaviours cancel in the energy-momentum tensor is predictable on the same grounds. As shown in [23], these terms stand for a vacuum generation of the gravitational waves. Eq. (2.36) is also a consequence of the asymptotic flatness. For the correct behaviour at $\mathcal{I}^+$, $T^{\mu\nu}_{\rm vac}$ *must* have the form (2.36). Finally, the structure of the vertex terms in (2.39)-(2.44) should, as the results below suggest, be a consequence of the condition of unitarity encoded in the one-loop triangular diagrams.

The problem of quantum noise.

The vacuum energy-momentum tensor $T^{\mu\nu}_{\rm vac}$ does not obey the dominant energy condition. Therefore, the energy flux $\Big(-dM/du\Big)$ in (1.36) is not positive definite but the total radiation energy in (1.37) is positive. The point here is that the quantity in (1.36) is an expectation value over a quantum state rather than a classical observable. The indefinite oscillations in $\Big(-dM/du\Big)$ should be within the quantum uncertainty, i.e. they represent a quantum noise which is present in $T^{\mu\nu}_{\rm vac}$ even at the asymptotically flat infinity $\mathcal{I}^+$ (see [22] for a detailed discussion). The problem is in discovering the mechanism by which the quantum noise gets averaged and the positive total energy emerges.

For the effective action in two dimensions [28]

$$S_{\rm vac} = {\rm const.} \int d^2x g^{1/2} R \, \frac{1}{\Box} \, R \tag{2.46}$$

this problem has a simple solution [5]. The positive radiation energy comes from the variation of the second-order form factor $1/\Box$ in (2.46). The contribution of this variation to $T^{\mu\nu}_{\rm vac}$ is quadratic in R and energy-dominant whereas the contribution of the variation δR is linear in R and sign indefinite. However, this latter contribution has the form of a total derivative and vanishes in the full integral over time [5]. This is the mechanism by which the quantum noise sums to zero for the whole history. A counterpart of Eq. (2.46) in four dimensions is [15]

$$S_{\rm vac} = {\rm const.} \int d^4x g^{1/2} \Re \log(-\Box) \Re \;\; + \; O[\Re^3] \quad , \tag{2.47}$$

and in this case the variation of the second-order form factor, $\log(-\Box)$, makes no contribution to $T^{\mu\nu}_{\rm vac}$ at $\mathcal{I}^+$ [23]. The apparent problem in four dimensions is that $T^{\mu\nu}_{\rm vac}\Big|_{\mathcal{I}^+}$ is entirely a total derivative, Eq. (2.36).

The insertion of (2.36) into (1.36) yields

$$-\frac{dM(u)}{du} = \frac{1}{(4\pi)^2} \, \frac{d^2}{du^2} \int d^2\mathcal{S}(\phi) \Big(r^2 \log(-\Box) I \Big) \Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} \quad . \tag{2.48}$$

Since this is a total derivative, the integrated energy flux is determined only by the limits of late time and early time $u = \pm\infty$. By the retardation property of the form factors , the contribution of early time vanishes (see Sec.5 for the proof). There remains only the contribution of late time:

$$M(-\infty) - M(\infty) = \lim_{u\to\infty} \frac{1}{(4\pi)^2} \, \frac{d}{du} \int d^2\mathcal{S}(\phi) \Big(r^2 \log(-\Box) I \Big) \Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} \quad , \tag{2.49}$$

and, for it to be finite and nonvanishing, the integrand in (2.49) should have a linear growth at late time

$$\Big(r^2 \log(-\Box) I \Big) \Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} \propto u \quad , \quad u \to \infty \quad . \tag{2.50}$$

Since the setting of the problem with external fields assumes that these fields become asymptotically stationary as $u \to \pm\infty$ [1], the growth in time required in (2.50) puzzles.

As we shall show, there can be no such growth if I in (2.50) is local in the field strength. The I_1 in Eq. (2.38) is local, and, therefore, the vacuum radiation of first order in $\Re$ is pure quantum noise [21,22]. The I_2 in Eq. (2.39) is quadratic in $\Re$ and nonlocal but the contribution of the trees is also pure quantum noise. The solution of the problem in four dimensions is that the growth in time is provided by the vertex operator (2.44). The kernel of this operator grows like u^{m+n-3}, $u \to \infty$, and the highest exponents m and n in all terms of (2.39) are precisely such that the result is (2.50). If the original quantum field contains no ghosts, the proportionality coefficient in (2.50) is positive.

3 Retarded kernels of the nonlocal operators

The retarded resolvent.

With the form factors in the spectral forms, the only nonlocal operator in the current (2.21) is the resolvent $1/(\Box - m^2)$ which, in the case of the expectation-value equations, is the retarded Green function [14]. It admits an expansion of the same nature as the action (2.2), i.e. the expansion up to terms $O[\Re^n]$ by covariant perturbation theory. To lowest order in $\Re$, the retarded operator $1/(\Box - m^2)$ acting on an arbitrary tensor $X^{\mu_1 \cdots \mu_n}$ is of the form

$$-\frac{1}{(\Box - m^2)} X^{\mu_1 \cdots \mu_n}(x) \tag{3.1}$$
$$= \frac{1}{4\pi} \int\limits_{\text{past of } x} d\bar{x} \bar{g}^{1/2} \Big(\delta(\sigma) - \theta(-\sigma) \, \frac{m J_1(m\sqrt{-2\sigma})}{\sqrt{-2\sigma}} \Big) g^{\mu_1}{}_{\bar{\mu}_1} \ldots g^{\mu_n}{}_{\bar{\mu}_n} X^{\bar{\mu}_1 \cdots \bar{\mu}_n}(\bar{x}) + O[\Re \times X]$$

where $\sigma = \sigma(x, \bar{x})$ is the world function [29,30] (one half of the square of the geodetic distance between x and $\bar{x}$), $g^{\mu}{}_{\bar{\mu}} = g^{\mu}{}_{\bar{\mu}}(x, \bar{x})$ is the propagator of the geodetic parallel transport for a vector [30], J_1 is the order-1 Bessel function, and the integration point $\bar{x}$ is in the past of the observation point x. Here and below, the bar over a symbol means that this symbol refers to the point $\bar{x}$.

The geometrical two-point functions entering the Green function (3.1) satisfy the equations [30]

$$g^{\mu\nu} \nabla_\mu \sigma \nabla_\nu \sigma = 2\sigma \quad , \quad \bar{g}^{\bar{\mu}\bar{\nu}} \bar{\nabla}_{\bar{\mu}} \sigma \bar{\nabla}_{\bar{\nu}} \sigma = 2\sigma \quad , \tag{3.2}$$

$$\sigma^\mu \nabla_\mu g^{\alpha}{}_{\bar{\alpha}} = 0 \quad , \quad \sigma^{\bar{\mu}} \bar{\nabla}_{\bar{\mu}} g^{\alpha}{}_{\bar{\alpha}} = 0 \quad , \quad g^{\alpha}{}_{\bar{\alpha}} \Big|_{x=\bar{x}} = \delta^{\alpha}{}_{\bar{\alpha}} \quad , \tag{3.3}$$

$$\sigma^\mu = -g^{\mu}{}_{\bar{\mu}} \, \sigma^{\bar{\mu}} \tag{3.4}$$

with

$$\sigma^\mu = g^{\mu\nu} \sigma_\nu \quad , \quad \sigma^{\bar{\mu}} = \bar{g}^{\bar{\mu}\bar{\nu}} \sigma_{\bar{\nu}} \quad , \quad \sigma_\nu = \nabla_\nu \sigma \quad , \quad \sigma_{\bar{\nu}} = \bar{\nabla}_{\bar{\nu}} \sigma \quad . \tag{3.5}$$

To lowest order in $\Re$ one can use the relations

$$g^{\mu}{}_{\bar{\mu}} = -\nabla^\mu \bar{\nabla}_{\bar{\mu}} \sigma + O[\Re] \quad , \tag{3.6}$$

$$\nabla_\alpha g^{\mu}{}_{\bar{\mu}} = O[\Re] \quad , \quad \bar{\nabla}_{\bar{\alpha}} g^{\mu}{}_{\bar{\mu}} = O[\Re] \quad . \tag{3.7}$$

As seen from (3.1), we are always dealing with some scalar source

$$\bar{X} = g^{\mu_1}_{\ \bar{\mu}_1}(x,\bar{x})\dots g^{\mu_n}_{\ \bar{\mu}_n}(x,\bar{x})X^{\bar{\mu}_1\dots\bar{\mu}_n}(\bar{x}) \tag{3.8}$$

which may depend parametrically on the observation point. Since the resolvent will be used only in the approximation (3.1) [8] , we shall omit the symbol $O[\Re\times X]$ and use the short notation

$$-\frac{1}{(\Box - m^2)}X(x) = \frac{1}{4\pi}\int\limits_{\text{past of }x} d\bar{x}\bar{g}^{1/2}\Big(\delta(\sigma) - \theta(-\sigma)\frac{mJ_1(m\sqrt{-2\sigma})}{\sqrt{-2\sigma}}\Big)\bar{X} \tag{3.9}$$

which will always assume that the source on the right-hand side is transported in a parallel fashion to the observation point.

For the integration over masses in the spectral integrals, the Green function should first be transformed by using the relation for the Bessel functions

$$\bar{\xi}^\mu\bar{\nabla}_\mu J_0(m\sqrt{-2\sigma}) = \frac{mJ_1(m\sqrt{-2\sigma})}{\sqrt{-2\sigma}}(\bar{\xi}\cdot\bar{\nabla}\sigma) \tag{3.10}$$

with an arbitrary timelike vector field ξ^μ, and the Gauss theorem in the form valid for null boundaries:

$$\int\limits_{\Omega} dx g^{1/2}\nabla_\mu A^\mu = \int dx g^{1/2}A^\mu\Big(\nabla_\mu\Sigma\Big)\delta(\Sigma) \quad . \tag{3.11}$$

Here Ω is an integration domain, $\Sigma = 0$ is the equation of its boundary, and $\Sigma < 0$ holds inside Ω. Integrating by parts in (3.9) one obtains the expression

$$-\frac{1}{(\Box - m^2)}X(x) = \frac{1}{4\pi}\int\limits_{\text{past of }x} d\bar{x}\bar{g}^{1/2}\theta(-\sigma)J_0(m\sqrt{-2\sigma})\bar{\nabla}_\alpha\Big(\frac{\bar{\xi}^\alpha}{(\bar{\xi}\cdot\bar{\nabla}\sigma)}\bar{X}\Big) \tag{3.12}$$

which holds with any choice of ξ^α, and from which the massless contribution proportional to $\delta(\sigma)$ is absent.

The kernels of $1/\Box$, $\log(-\Box)$, and of the vertex operators.

[8] The curvature correction to the lowest-order form factor is calculated in Sec.4.

The retarded kernel of the massless operator $1/\Box$ is given by Eq. (3.9) with $m = 0$. Up to $O[\Re \times X]$,

$$-\frac{1}{\Box}X(x) = \frac{1}{4\pi}\int\limits_{\text{past of } x} d\bar{x}\bar{g}^{1/2}\delta\Big(\sigma(x,\bar{x})\Big)\bar{X} \tag{3.13}$$

where the integration is over the past light cone of the observation point x .

The retarded kernel of the operator $\log(-\Box)$ has been calculated in [21,22]. To lowest order in $\Re$ it is of the form

$$\log(-\Box)X(x) = \frac{1}{2\pi}\int\limits_{\text{past of } x} d\bar{x}\bar{g}^{1/2}\delta'\Big(\sigma(x,\bar{x})\Big)\bar{X} + \text{ a local term} \tag{3.14}$$

where the local term, i.e. a term proportional to X at the observation point x, is irrelevant to the present consideration [9].

The kernels of the vertex operators (2.44) are obtained by using the following spectral form of their generating function:

$$-\frac{\log(j_1\Box_1/j_2\Box_2)}{j_1\Box_1 - j_2\Box_2} = \frac{1}{j_1 j_2}\int\limits_0^\infty \frac{dm^2}{(m^2/j_1 - \Box_1)(m^2/j_2 - \Box_2)} \quad . \tag{3.15}$$

Combining (3.15) and (3.12) one finds

$$\begin{aligned} -\frac{\log(j_1\Box_1/j_2\Box_2)}{j_1\Box_1 - j_2\Box_2}X_1X_2(x) \quad &= \quad \frac{1}{4\pi}\int\limits_{\text{past of } x} d\bar{x}_1\bar{g}_1^{1/2}\theta(-\sigma_1)\bar{\nabla}_\alpha\Big(\frac{\bar{\xi}^\alpha}{(\bar{\xi}\cdot\bar{\nabla}\sigma_1)}\bar{X}_1\Big) \\ &\times \quad \frac{1}{4\pi}\int\limits_{\text{past of } x} d\bar{x}_2\bar{g}_2^{1/2}\theta(-\sigma_2)\bar{\nabla}_\alpha\Big(\frac{\bar{\xi}^\alpha}{(\bar{\xi}\cdot\bar{\nabla}\sigma_2)}\bar{X}_2\Big) \\ &\times \quad \frac{1}{j_1 j_2}\int\limits_0^\infty dm^2 J_0(m\sqrt{-2\sigma_1/j_1})J_0(m\sqrt{-2\sigma_2/j_2}) \end{aligned} \tag{3.16}$$

where

$$\sigma_1 = \sigma(x,\ \bar{x}_1) \quad , \quad \sigma_2 = \sigma(x,\ \bar{x}_2) \quad . \tag{3.17}$$

The spectral-mass integral in (3.16) can be calculated with the aid of the Fourier-Bessel relation

$$\int\limits_0^\infty dm^2 J_0(m\sqrt{-2\sigma_1/j_1})J_0(m\sqrt{-2\sigma_2/j_2}) = 2\delta\Big(\frac{\sigma_1}{j_1} - \frac{\sigma_2}{j_2}\Big) \quad . \tag{3.18}$$

[9] Expression (3.14) without a further specification is valid only for x located outside the support of X. This includes the case where $x \to \mathcal{I}^+$ provided that $X\Big|_{\mathcal{I}^+} = O(1/r^3)$. The latter case is the one that we presently consider. In the general case, the integral on the right-hand side of (3.14) is improper. For its precise definition and the form of the local term see [21,22].

It is convenient to factorize the vertex function by introducing an auxiliary integration over a parameter. We write

$$\theta(-\sigma_1)\theta(-\sigma_2)\delta\Big(\frac{\sigma_1}{j_1}-\frac{\sigma_2}{j_2}\Big)=\int\limits_{-\infty}^{0} dq\;\delta\Big(q-\frac{\sigma_1}{j_1}\Big)\delta\Big(q-\frac{\sigma_2}{j_2}\Big)\quad , \tag{3.19}$$

and introduce the operator $\mathcal{H}_q$ depending on the parameter q and defined as follows:

$$\mathcal{H}_q X(x)=\frac{1}{4\pi}\int\limits_{\text{past of }x} d\bar{x}\bar{g}^{1/2}\delta\Big(\sigma(x,\bar{x})-q\Big)\bar{X}\quad ,\quad q\leq 0\quad . \tag{3.20}$$

Here the integration is over the past sheet of the hyperboloid $\sigma(x,\bar{x})=q$ associated with the point x (the past hyperboloid of x , Fig.3). The operator $\mathcal{H}_q$ is a generalization of the retarded operator $-1/\Box$, Eq. (3.13). A calculation of the derivative

$$\begin{aligned}\frac{d}{dq}\mathcal{H}_q X(x) &= \frac{1}{4\pi}\int\limits_{\text{past of }x} d\bar{x}\bar{g}^{1/2}\delta'\Big(q-\sigma(x,\bar{x})\Big)\bar{X} \\ &= \frac{1}{4\pi}\int\limits_{\text{past of }x} d\bar{x}\bar{g}^{1/2}\delta\Big(\sigma(x,\bar{x})-q\Big)\bar{\nabla}_\alpha\Big(\frac{\bar{\xi}^\alpha}{(\bar{\xi}\cdot\bar{\nabla}\sigma)}\bar{X}\Big)\end{aligned} \tag{3.21}$$

with the result containing an arbitrary timelike vector field ξ^α shows that this derivative with q replaced by jq will appear in (3.16) upon the use of (3.18) and (3.19).

In this way we obtain

$$-\frac{\log(j_1\Box_1/j_2\Box_2)}{j_1\Box_1-j_2\Box_2}\,X_1X_2(x)=2\int\limits_{-\infty}^{0} dq\Big[\frac{d}{d(qj_1)}\mathcal{H}_{qj_1}X_1(x)\Big]\Big[\frac{d}{d(qj_2)}\mathcal{H}_{qj_2}X_2(x)\Big]\quad . \tag{3.22}$$

Finally, the result for the kernel of the vertex operator (2.44) is

$$F(m,n)X_1X_2(x)=-2\int\limits_{-\infty}^{0} dq q^{m+n}\Big[\Big(\frac{d}{dq}\Big)^{m+1}\mathcal{H}_q X_1(x)\Big]\Big[\Big(\frac{d}{dq}\Big)^{n+1}\mathcal{H}_q X_2(x)\Big]\quad . \tag{3.23}$$

The kernels of the vertex operators superposed with $1/\Box$.

There are two more types of the vertex operators in (2.39):

$$(a)\quad \frac{1}{\Box_2}\,F(m,n)\quad ,\quad (b)\quad \frac{1}{\Box_1\Box_2}\,F(m,n)\quad . \tag{3.24}$$

It is important that in (a) $n \geq 1$, and in (b) $m \geq 1$, $n \geq 1$, i.e. the appearance of a factor $1/\Box_1$ or $1/\Box_2$ in the vertex is necessarily accompanied by the appearance of a derivative $\partial/\partial j$ bearing the same number [10]. Therefore, it suffices to consider the generating expressions

$$-\frac{1}{\Box_2}\frac{\partial}{\partial j_2}\frac{\log(j_1\Box_1/j_2\Box_2)}{j_1\Box_1 - j_2\Box_2}X_1X_2 = \frac{1}{j_1 {j_2}^2}\int\limits_0^\infty \frac{dm^2}{(m^2/j_1 - \Box_1)(m^2/j_2 - \Box_2)^2}X_1X_2\ , \quad (3.25)$$

$$-\frac{1}{\Box_1\Box_2}\frac{\partial}{\partial j_1}\frac{\partial}{\partial j_2}\frac{\log(j_1\Box_1/j_2\Box_2)}{j_1\Box_1 - j_2\Box_2}X_1X_2 = \frac{1}{{j_1}^2 {j_2}^2}\int\limits_0^\infty \frac{dm^2}{(m^2/j_1 - \Box_1)^2(m^2/j_2 - \Box_2)^2}X_1X_2\ . \quad (3.26)$$

The square (and, generally, any power) of the massive Green function is a well defined operator whose kernel can be obtained by differentiating the Green function with respect to the mass:

$$\frac{1}{(m^2 - \Box)^2}X = -\frac{\partial}{\partial m^2}\frac{1}{(m^2 - \Box)}X \quad . \quad (3.27)$$

The behaviour of the function (3.27) as $m^2 \to 0$ and hence the convergence of the integrals (3.25),(3.26) at the lower limits depends on the class of the sources X. This question will be considered in Sec.5.

The differentiation of the Green function with respect to the mass is readily accomplished in the original expression (3.9) by using the relation for the Bessel functions

$$\frac{\partial}{\partial m^2}\ \frac{mJ_1(m\sqrt{-2\sigma})}{\sqrt{-2\sigma}} = \frac{1}{2}\ J_0(m\sqrt{-2\sigma}) \quad . \quad (3.28)$$

The result is

$$\frac{1}{(m^2 - \Box)^2}X(x) = \frac{1}{8\pi}\int\limits_{\text{past of } x} d\bar{x}\bar{g}^{1/2}\theta(-\sigma)J_0(m\sqrt{-2\sigma})\bar{X} \quad . \quad (3.29)$$

The insertion of (3.29) and (3.12) into (3.25), (3.26) leads to the same spectral integral and the same factorization procedure as in the previous case. For the generating expressions we obtain

$$-\frac{1}{\Box_2}\frac{\partial}{\partial j_2}\frac{\log(j_1\Box_1/j_2\Box_2)}{j_1\Box_1 - j_2\Box_2}\ X_1X_2(x) = \int\limits_{-\infty}^0 dq\Big[\frac{d}{d(qj_1)}\mathcal{H}_{qj_1}X_1(x)\Big]\Big[\frac{1}{j_2}\mathcal{H}_{qj_2}X_2(x)\Big]\ , \quad (3.30)$$

[10] This is a manifestation of the general property of the form factors established in [17] with the aid of the integral α-representation and called there "rule of the like α".

$$-\frac{1}{\Box_1\Box_2}\frac{\partial}{\partial j_1}\frac{\partial}{\partial j_2}\frac{\log(j_1\Box_1/j_2\Box_2)}{j_1\Box_1-j_2\Box_2}\ X_1X_2(x)=\frac{1}{2}\int\limits_{-\infty}^{0}dq\Big[\frac{1}{j_1}\mathcal{H}_{qj_1}X_1(x)\Big]\Big[\frac{1}{j_2}\mathcal{H}_{qj_2}X_2(x)\Big]\ , \tag{3.31}$$

and, finally, the results for the kernels of the vertex operators are

$$\frac{1}{\Box_2}F(m,n)X_1X_2(x) \tag{3.32}$$

$$=-\int\limits_{-\infty}^{0}dqq^{m+n}\Big[\Big(\frac{d}{dq}\Big)^{m+1}\mathcal{H}_qX_1(x)\Big]\Big[\Big(\frac{d}{dq}\Big)^{n-1}\frac{1}{q}\mathcal{H}_qX_2(x)\Big]\ ,\quad n\geq 1,$$

$$\frac{1}{\Box_1\Box_2}F(m,n)X_1X_2(x) \tag{3.33}$$

$$=-\frac{1}{2}\int\limits_{-\infty}^{0}dqq^{m+n}\Big[\Big(\frac{d}{dq}\Big)^{m-1}\frac{1}{q}\mathcal{H}_qX_1(x)\Big]\Big[\Big(\frac{d}{dq}\Big)^{n-1}\frac{1}{q}\mathcal{H}_qX_2(x)\Big]\ ,\quad m\geq 1,\ n\geq 1.$$

The question of convergence of the integrals (3.25),(3.26) at $m^2=0$ transfers now to the integrals (3.32),(3.33) with respect to q whose convergence at $q=-\infty$ should be provided by the properties of the sources X.

4 The asymptotic behaviours at $\mathcal{I}^+$

Null hyperplanes.

The power of growth of the world function as one of its points tends to $\mathcal{I}^+$ and the other one stays in a compact domain should be the same as in flat spacetime. Therefore, we may write

$$\sigma(x,\ \bar{x})\Big|_{x\in\mathcal{I}^+[u,\phi,r\to\infty]} = rZ(u,\phi\,;\ \bar{x}) + O(r^0) \tag{4.1}$$

with some coefficient function Z.

Inserting the behaviour (4.1) into the equation (3.2) for σ with respect to the point x and using the asymptotic form of the metric at $\mathcal{I}^+$ we obtain

$$-2rZ\frac{\partial}{\partial u}Z + O(r^0) = 2rZ + O(r^0) \tag{4.2}$$

whence

$$\frac{\partial}{\partial u}Z(u,\phi\,;\ \bar{x}) = -1 \quad . \tag{4.3}$$

Therefore, we introduce a new notation to write down the solution for Z

$$Z(u,\phi\,;\ \bar{x}) = -u + U_\phi(\bar{x}) \quad , \tag{4.4}$$

and rewrite Eq. (4.1) in its final form

$$\sigma(x,\ \bar{x})\Big|_{x\in\mathcal{I}^+[u,\phi,r\to\infty]} = r\Big(-u + U_\phi(\bar{x})\Big) + O(r^0) \quad . \tag{4.5}$$

Inserting the behaviour (4.5) into the equation (3.2) for σ with respect to the point $\bar{x}$ we obtain

$$\Big(\bar{\nabla}U(\bar{x})\Big)^2 = 0 \quad , \quad U(\bar{x}) \equiv U_\phi(\bar{x}) \quad . \tag{4.6}$$

Hence we infer that the equation

$$U_\phi(\bar{x}) = u \tag{4.7}$$

with fixed ϕ defines a family of null hypersurfaces labelled by the retarded time u but different from the future light cones of the Bondi-Sachs frame. The hypersurfaces (4.7) will

be called hyperplanes [11] . To different directions ϕ at infinity there correspond different families of null hyperplanes.

An important property of the function $U_\phi(\bar{x})$ follows from Eq. (3.4). Inserting the behaviour (4.5) in (3.4) one obtains

$$\bar{\nabla}_{\bar{\mu}} U_\phi(\bar{x}) = g_{\bar{\mu}}{}^{\mu}(\bar{x},x)\nabla_\mu u - g_{\bar{\mu}}{}^{\mu}(\bar{x},x)\nabla_\mu \phi^a \frac{\partial}{\partial \phi^a} U_\phi(\bar{x}) \tag{4.8}$$

where $a = 1, 2$, and $x \to \mathcal{I}^+$. By (1.18),

$$\nabla_\mu \phi \quad \propto \quad \frac{1}{r} m_\mu \Big|_{\mathcal{I}^+} \tag{4.9}$$

while the contraction $g_{\bar{\mu}}{}^{\mu}(\bar{x},x)m_\mu$ remains finite as $x \to \mathcal{I}^+$ [23]. Therefore,

$$g_{\bar{\mu}}{}^{\mu}(\bar{x},x)\nabla_\mu \phi^a \Big|_{x\to\mathcal{I}^+} = O\Big(\frac{1}{r}\Big) \quad . \tag{4.10}$$

As a result, one obtains the following law of parallel transport:

$$\bar{\nabla}_{\bar{\mu}} U_\phi(\bar{x}) = g_{\bar{\mu}}{}^{\mu}(\bar{x},x)\nabla_\mu u(x)\Big|_{x\to\mathcal{I}^+[u,\phi]} \quad . \tag{4.11}$$

The geometric meaning of the equations above will clarify if one answers the following question: what becomes of the past light cone of a point x at the limit $x \to \mathcal{I}^+$? At this limit, one of the null generators of the cone merges with $\mathcal{I}^+$ i.e. disappears from any compact domain. Therefore, the resultant limiting surface is no more a cone although it remains a null hypersurface. As follows from Eq. (4.5), this limiting surface is none other than the hyperplane (4.7) whose parameters u, ϕ label the point of $\mathcal{I}^+$ to which the vertex of the prelimiting cone comes (Fig.4). The null generators of the prelimiting cone all but one become the generators of the limiting hyperplane. It follows that the generators of each hyperplane in (4.7) are the null geodesics that, when traced towards the future, come to one and the same point ϕ of the celestial sphere at one and the same instant u of retarded time. This can be taken for a definition of light rays parallel in the future. Since the null generators of a hyperplane merge at infinity, they have a common tangent vector at $\mathcal{I}^+$ which is ∇u. On the other hand, at a point $\bar{x}$ of a compact domain

[11] More specifically, they should have been called *future* hyperplanes as distinct from the *past* hyperplanes defined similarly by the conditions at $\mathcal{I}^-$. However, the past hyperplanes do not figure in the present consideration.

the vector tangent to the generator is $\bar{\nabla}_{\bar{\mu}}U_\phi(\bar{x})$. Eq. (4.11) is, therefore, the law of parallel transport of the tangent vector along the null geodesic emanating from a given point and belonging to a given hyperplane.

Further properties of the function $U_\phi(\bar{x})$ derive under the assumption that for both the point $\bar{x}$ and the point at $\mathcal{I}^+$ one can use one and the same global Bondi-Sachs frame:

$$U_\phi(\bar{x}) = U_\phi(\bar{u}, \bar{\phi}, \bar{r}) \quad . \tag{4.12}$$

Then we have

$$U_\phi(\bar{u}, \bar{\phi}, \bar{r}) \geq \bar{u} \quad , \tag{4.13}$$

and

$$U_\phi(\bar{u}, \bar{\phi}, \bar{r})\Big|_{\bar{\phi}=\phi} = \bar{u} \quad , \quad \frac{\partial}{\partial\bar{\phi}}U_\phi(\bar{u}, \bar{\phi}, \bar{r})\Big|_{\bar{\phi}=\phi} = 0 \quad . \tag{4.14}$$

The inequality (4.13) is a consequence of the general fact mentioned in Sec.1 that, for the points $\bar{x}$ belonging to the past light cone of x , $u(\bar{x}) \leq u(x)$. With x at $\mathcal{I}^+$, the latter inequality holds for the points $\bar{x}$ belonging to the limiting hyperplane (4.7). Eqs. (4.14) follow from the fact that the radial geodesic $\bar{u} = u$, $\bar{\phi} = \phi$ along which the point x in (4.5) tends to $\mathcal{I}^+$ serves at the same time as a generator of the past light cone of x and, therefore, belongs to the limiting hyperplane (4.7). The second equation in (4.14) implies $\bar{\nabla}U_\phi\Big|_{\bar{\phi}=\phi} = \bar{\nabla}\bar{u}$ and follows from the first equation and Eq. (4.6).

Finally, in the case where $\bar{x}$ is at the future asymptotically flat infinity $\mathcal{I}^+$ or i^+, one can use the flat-spacetime formula for $U_\phi(\bar{x})$. Indeed, in this case the geodesic connecting the points x and $\bar{x}$ in (4.5) passes entirely through the asymptotically flat domain.[12] One has

$$\begin{aligned} U_\phi(\bar{x})\Big|_{\bar{x}\to\mathcal{I}^+, i^+} &= \bar{u} + \bar{r}\Big(1 - \cos\omega(\phi,\ \bar{\phi})\Big) \\ &= \bar{t} - n_i(\phi)\bar{\mathbf{x}}^i \end{aligned} \tag{4.15}$$

where the first form refers to the Bondi-Sachs parametrization (4.12) and the second to the Minkowski coordinates (1.25) for $\bar{x}$. In (4.15), $\omega(\phi,\ \bar{\phi})$ is the arc length between the

[12]The geometric two-point functions $\sigma(x,\ \bar{x})$ and $g^\mu{}_{\bar{\mu}}(x,\ \bar{x})$ are nonlocal objects involving the metric on the geodesic connecting the two points. Therefore, even if both points are at the asymptotically flat infinity but one is in the future and the other in the past, the flat-spacetime approximation is invalid since the geodesic connecting the two points passes through a domain of strong field.

points ϕ and $\bar{\phi}$ on the unit 2-sphere:

$$\cos\omega(\phi,\ \bar{\phi}) = n_i(\phi)n^i(\bar{\phi}) \quad . \tag{4.16}$$

Here $n_i(\phi)$ is the direction vector introduced in (1.26).

The $1/\Box$ and $\log(-\Box)$ at $\mathcal{I}^+$.

The behaviours of the kernels of $1/\Box$ and $\log(-\Box)$ at $\mathcal{I}^+$ follow immediately from (3.13),(3.14) and (4.5):

$$\frac{1}{\Box}X\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} = -\frac{1}{r}D_{\mathbf{1}}(u,\phi|X) \quad , \tag{4.17}$$

$$\log(-\Box)X\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} = -\frac{2}{r^2}\frac{\partial}{\partial u}D_{\mathbf{1}}(u,\phi|X) \quad , \tag{4.18}$$

and the coefficient of these behaviours is an integral over the null hyperplane (4.7):

$$D_{\mathbf{1}}(u,\phi|X) = \frac{1}{4\pi}\int d\bar{x}\bar{g}^{1/2}\delta\Big(U_\phi(\bar{x}) - u\Big)\bar{X} \quad . \tag{4.19}$$

As discussed in Sec.7 below, this integral is the ultrarelativistic limiting case of a more general hypersurface integral D called radiation moment of the source X. The subscript $\mathbf{1}$ that $D_{\mathbf{1}}$ in (4.19) bears is to distinguish the ultrarelativistic moment.

It follows from Eqs. (4.18) and (2.48) that the moment $D_{\mathbf{1}}$ determines the flux of vacuum energy through $\mathcal{I}^+$. We obtain

$$\frac{dM(u)}{du} = \frac{2}{(4\pi)^2}\,\frac{\partial^3}{\partial u^3}\,\int d^2\mathcal{S}(\phi)\,D_{\mathbf{1}}(u,\phi|I) \tag{4.20}$$

where I is the scalar in (2.36). Since $D_{\mathbf{1}}$ serves at the same time as a coefficient in (4.17), it governs also the classical radiation (see, e.g., Sec.8 below). Of special importance is, therefore, the question of convergence of the integral (4.19). It is only with this convergence that the behaviours (4.17) and (4.18) hold.

Note that the convergence of the original integral (3.13) defining the retarded Green function imposes a restriction only on the behaviour of the source X at the past null infinity whereas the convergence of the moment (4.19) requires certain behaviours of X

also at the future null infinity and spatial infinity. The behaviour at the future null infinity is critical.

To study the convergence of the integral (4.19) at $\mathcal{I}^+$ consider a portion of the integration domain defined by the following inequalities in the Bondi-Sachs frame:

$$\Omega \quad : \quad u_2 > \bar{u} > u_1 \quad , \quad \phi + \Delta > \bar{\phi} > \phi - \Delta \quad , \quad \bar{r} > r_0 \tag{4.21}$$

with large r_0, small Δ, and any fixed u_1, u_2. In the integral restricted to Ω, the integration point $\bar{x}$ is near $\mathcal{I}^+$. Therefore, one can use the flat-spacetime expressions for the metric and the function $U_\phi(\bar{x})$, Eqs. (1.15) and (4.15). One finds

$$\begin{aligned}\int\limits_\Omega d\bar{x}\bar{g}^{1/2}\delta\Big(U_\phi(\bar{x}) - u\Big)\bar{X} \;&=\; \int\limits_{u_1}^{u_2} d\bar{u} \int\limits_{\phi-\Delta}^{\phi+\Delta} d^2\mathcal{S}(\bar{\phi}) \int\limits_{r_0}^{\infty} d\bar{r}\,\bar{r}^2\delta\Big(\bar{u} - u + \bar{r}(1-\cos\omega)\Big)\bar{X} \\ &=\; \int\limits_{u_1}^{u_2} d\bar{u} \int\limits_{\phi-\Delta}^{\phi+\Delta} \frac{d^2\mathcal{S}(\bar{\phi})}{1-\cos\omega}\Big(\bar{r}^2\bar{X}\Big)_{\mathcal{I}^+}[\bar{u},\bar{\phi},\bar{r} = \frac{u-\bar{u}}{1-\cos\omega}]\end{aligned} \tag{4.22}$$

with $\omega = \omega(\phi, \bar{\phi})$ in (4.16), and $\bar{r}$ large i.e. $(1 - \cos\omega)$ small. In agreement with the fact that the null hyperplane reaches the future infinity at only one value of $\bar{\phi}$, $\bar{\phi} = \phi$, the question of convergence concerns the angle integral. The source $\bar{X}$ may have any power of decrease or even increase at $\mathcal{I}^+$

$$\bar{X}\Big|_{\mathcal{I}^+[\bar{u},\bar{\phi},\bar{r}\to\infty]} = \kappa(\bar{u},\bar{\phi})\bar{r}^n \tag{4.23}$$

provided that

$$\kappa(\bar{u},\bar{\phi})\Big|_{\bar{\phi}\to\phi} = O\Big((\bar{\phi}-\phi)^{5+2n}\Big) \quad . \tag{4.24}$$

However, to have $5+2n \neq 0$ in (4.24), the source $\bar{X}$ should depend on the external angles ϕ. This is possible in the case of a tensor X since in this case $\bar{X}$ depends parametrically on the observation point through the propagators of parallel transport, Eq. (3.8). For a scalar source that has no special relation to the integration hyperplane in (4.19), the convergence condition is $5 + 2n = 0$, or, with the analyticity of X at $\mathcal{I}^+$,

$$X\Big|_{\mathcal{I}^+} = O\Big(\frac{1}{r^3}\Big) \quad . \tag{4.25}$$

The I in Eqs. (2.36)-(2.39) and (4.20) is a scalar. We must, therefore, make sure that it satisfies the criterion (4.25):

$$I\Big|_{\mathcal{I}^+} = O\Big(\frac{1}{r^3}\Big) \quad . \tag{4.26}$$

Let us check if this criterion is fulfilled for the vertex contributions.

The vertex operators at $\mathcal{I}^+$.

The asymptotic behaviours of the vertex functions (3.23) as $x \to \mathcal{I}^+$ are obtained by making the following replacement of the integration variable in (3.23):

$$q = r(\tau - u) \tag{4.27}$$

where τ is the new integration variable, and u, r are the parameters of

$$x \in \mathcal{I}^+[u, \phi, r \to \infty] \quad .$$

With q replaced as in (4.27), and τ fixed,

$$\mathcal{H}_q X(x)\Big|_{x\in\mathcal{I}^+[u,\phi,r\to\infty]} = \frac{1}{r} D_{\mathbf{1}}(\tau, \phi|X) \quad . \tag{4.28}$$

As a result we obtain

$$\begin{aligned} &F(m,n)X_1X_2\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} \\ &= -\frac{2}{r^3}\int\limits_{-\infty}^{u} d\tau(\tau-u)^{m+n}\Big[\Big(\frac{d}{d\tau}\Big)^{m+1} D_{\mathbf{1}}(\tau,\phi|X_1)\Big]\Big[\Big(\frac{d}{d\tau}\Big)^{n+1} D_{\mathbf{1}}(\tau,\phi|X_2)\Big] \, . \end{aligned} \tag{4.29}$$

In the same way, for the vertex functions (3.32) and (3.33) we obtain

$$\begin{aligned} &\frac{1}{\Box_2}F(m,n)X_1X_2\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} \\ &= -\frac{1}{r^2}\int\limits_{-\infty}^{u} d\tau(\tau-u)^{m+n}\Big[\Big(\frac{d}{d\tau}\Big)^{m+1} D_{\mathbf{1}}(\tau,\phi|X_1)\Big]\Big[\Big(\frac{d}{d\tau}\Big)^{n-1} \frac{1}{(\tau-u)} D_{\mathbf{1}}(\tau,\phi|X_2)\Big] \, , \end{aligned} \tag{4.30}$$

$$\begin{aligned} &\frac{1}{\Box_1\Box_2}F(m,n)X_1X_2\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} \\ &= -\frac{1}{2r}\int\limits_{-\infty}^{u} d\tau(\tau-u)^{m+n}\Big[\Big(\frac{d}{d\tau}\Big)^{m-1} \frac{1}{(\tau-u)} D_{\mathbf{1}}(\tau,\phi|X_1)\Big]\Big[\Big(\frac{d}{d\tau}\Big)^{n-1} \frac{1}{(\tau-u)} D_{\mathbf{1}}(\tau,\phi|X_2)\Big] \, . \end{aligned} \tag{4.31}$$

Thus we infer that, with the general sources, the operator in (4.29) does have the needed power of decrease at $\mathcal{I}^+$ but the operators in (4.30) and (4.31) do not. In Eqs.

(2.39)-(2.42) these latter operators act only on the vector and tensor sources and are accompanied by extra derivatives but the derivatives alone do not help since they have the projections tangential to $\mathcal{I}^+$. What helps is the conservation of the vector and tensor sources. Owing to this conservation, the tangential derivatives vanish, and the behaviours in (4.30) and (4.31) soften down by at least one power of $1/r$. As will be seen below, in the case of the vector vertex this is still insufficient and, to ensure condition (4.26), we shall have to impose a special limitation on the vector source.

The conserved charges.

Below, for the moments $D_{\mathbf{1}}$ of the sources in (1.8) we use the short notations

$$D_{\mathbf{1}}(\tau,\phi|\hat{P}) = \hat{D}_{\mathbf{1}}(\tau,\phi) = \hat{D}_{\mathbf{1}} \quad , \tag{4.32}$$

$$D_{\mathbf{1}}(\tau,\phi|\hat{J}^\mu) = \hat{D}_{\mathbf{1}}{}^\mu(\tau,\phi) = \hat{D}_{\mathbf{1}}{}^\mu \quad , \tag{4.33}$$

$$D_{\mathbf{1}}(\tau,\phi|J^{\mu\nu}) = D_{\mathbf{1}}{}^{\mu\nu}(\tau,\phi) = D_{\mathbf{1}}{}^{\mu\nu} \quad . \tag{4.34}$$

To distinguish the moments of the Ricci scalar R and the matrix $\hat{Q}$ in (1.12), they will be denoted

$$D_{\mathbf{1}}(\tau,\phi|R) = D_{\mathbf{1}}{}^R(\tau,\phi) = D_{\mathbf{1}}{}^R \quad , \tag{4.35}$$

$$D_{\mathbf{1}}(\tau,\phi|\hat{Q}) = \hat{D}_{\mathbf{1}}{}^Q(\tau,\phi) = \hat{D}_{\mathbf{1}}{}^Q \quad . \tag{4.36}$$

We have

$$g_{\mu\nu}D_{\mathbf{1}}{}^{\mu\nu} = -D_{\mathbf{1}}{}^R \quad . \tag{4.37}$$

The non-scalar moments contain the propagators of parallel transport. Specifically [13],

$$\hat{D}_{\mathbf{1}}{}^\mu = \frac{1}{4\pi}\int d\bar{x}\bar{g}^{1/2}\delta\Big(U_\phi(\bar{x})-\tau\Big)g^\mu{}_{\bar{\mu}}(x,\bar{x})\hat{J}^{\bar{\mu}}(\bar{x})\Big|_{x\to\mathcal{I}^+[\tau,\phi]} \quad , \tag{4.38}$$

$$D_{\mathbf{1}}{}^{\mu\nu} = \frac{1}{4\pi}\int d\bar{x}\bar{g}^{1/2}\delta\Big(U_\phi(\bar{x})-\tau\Big)g^\mu{}_{\bar{\mu}}(x,\bar{x})g^\nu{}_{\bar{\nu}}(x,\bar{x})J^{\bar{\mu}\bar{\nu}}(\bar{x})\Big|_{x\to\mathcal{I}^+[\tau,\phi]} \quad , \tag{4.39}$$

and it is important that the propagators $g^\alpha{}_{\bar{\alpha}}(x,\bar{x})$ are taken at the point x of $\mathcal{I}^+$ with the same parameters (τ,ϕ) as in the $\delta-$function. The moments $\hat{D}_{\mathbf{1}}, \hat{D}_{\mathbf{1}}{}^\alpha, D_{\mathbf{1}}{}^{\alpha\beta}$, etc. have the

[13] The matrix moments contain implicitly the propagators of parallel transport for the matrix indices.

transformation properties suggested by their indices but it should be remembered that they are defined only for a point at $\mathcal{I}^+$. Thus, the metric $g_{\mu\nu}$ in Eq. (4.37) can only be at the point (τ, ϕ) of $\mathcal{I}^+$. Eq. (4.37) itself is a consequence of the relation [30]

$$g_{\mu\nu}(x) g^{\mu}{}_{\bar{\mu}}(x, \bar{x}) g^{\nu}{}_{\bar{\nu}}(x, \bar{x}) = g_{\bar{\mu}\bar{\nu}}(\bar{x}) \quad . \tag{4.40}$$

Using the law of parallel transport in (4.11) we may calculate

$$\begin{aligned} \nabla_\alpha u \frac{d}{d\tau} \hat{D}_{\mathbf{1}}{}^{\alpha} &= -\frac{1}{4\pi} \int d\bar{x} \bar{g}^{1/2} \delta'\Big(U_\phi(\bar{x}) - \tau\Big) \bar{\nabla}_{\bar{\mu}} U_\phi(\bar{x}) \hat{J}^{\bar{\mu}}(\bar{x}) \\ &= -\frac{1}{4\pi} \int d\bar{x} \bar{g}^{1/2} \bar{\nabla}_{\bar{\mu}} \delta\Big(U_\phi(\bar{x}) - \tau\Big) \hat{J}^{\bar{\mu}}(\bar{x}) \quad . \end{aligned} \tag{4.41}$$

Since, by assumption, the support of $\hat{J}^\mu$ is confined to a spacetime tube, the integration by parts in (4.41) gives rise to no boundary term. Hence, using the conservation law (1.9) we obtain

$$\nabla_\alpha u \frac{d}{d\tau} \hat{D}_{\mathbf{1}}{}^{\alpha} = 0 \quad . \tag{4.42}$$

The conserved quantity in (4.42)

$$\hat{e} \equiv \nabla_\alpha u \, \hat{D}_{\mathbf{1}}{}^{\alpha} \tag{4.43}$$

is the full charge of the vector source since the charge can be written as the integral

$$\hat{e} = \frac{1}{4\pi} \int d\bar{x} \bar{g}^{1/2} \delta\Big(\Sigma(\bar{x})\Big) \bar{\nabla}_{\bar{\mu}} \Sigma \; \hat{J}^{\bar{\mu}}(\bar{x}) \tag{4.44}$$

over any hypersurface $\Sigma = 0$ crossing the support tube of $\hat{J}^\mu$ ($\nabla\Sigma$ is past directed). In expression (4.41) this hypersurface is a null hyperplane. If $\hat{J}^\mu$ is a pure vector like in the case of the electromagnetic current, Eqs. (4.42)-(4.44) are exact. In the general case where the integral (4.44) contains the matrix propagators of parallel transport, they are valid with accuracy $O[\Re^2]$ since the proof of conservation makes use of Eq. (3.7).

The significance of the assumption about the support of $\hat{J}^\mu$ is in the fact that it excludes a radiation of the charge. In the general case, the full conserved charge is given by Eq. (4.44) in which the hypersurface $\Sigma = 0$ is spacelike. With this hypersurface null, the charge is generally not conserved. It particular, if this hypersurface is the radial light cone $u =$ const., Eq. (4.44) defines the Bondi charge

$$\hat{e}(u) = \frac{1}{4\pi} \int d\bar{x} \bar{g}^{1/2} \delta\Big(\bar{u} - u\Big) \bar{\nabla}_{\bar{\mu}} \bar{u} \; \hat{J}^{\bar{\mu}}(\bar{x}) \quad . \tag{4.45}$$

Then we have by using Eq. (3.11) :

$$\begin{aligned}\frac{d}{du}\hat{e}(u) &= -\frac{1}{4\pi}\int d\bar{x}\bar{g}^{1/2}\bar{\nabla}_{\bar{\mu}}\delta\Big(\bar{u}-u\Big)\hat{J}^{\bar{\mu}}(\bar{x}) \\ &= -\lim_{v_0\to\infty}\frac{1}{4\pi}\int d\bar{x}\bar{g}^{1/2}\delta\Big(\bar{u}-u\Big)\delta\Big(\bar{v}-v_0\Big)\bar{\nabla}_{\bar{\mu}}\bar{v}\hat{J}^{\bar{\mu}}(\bar{x}) \qquad (4.46)\end{aligned}$$

with v in (1.17). The boundary $v = v_0$ with $v_0 \to \infty$ is $\mathcal{I}^+$. Since at this boundary one can use the asymptotic form of the Bondi-Sachs metric (1.15), the result is

$$-\frac{d\hat{e}(u)}{du} = \frac{1}{4\pi}\int d^2\mathcal{S}(\phi)\left(\frac{1}{2}r^2\nabla_\mu v\hat{J}^\mu\right)\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} \qquad (4.47)$$

which is an equation analogous to (1.35). If, on the other hand, the support of $\hat{J}^\mu$ is confined to a tube, we have $\hat{J}^\mu\Big|_{\mathcal{I}^+} \equiv 0$, and $d\hat{e}/du = 0$.

In the case of the tensor source, a calculation similar to (4.41)-(4.42) and valid with accuracy $O[\Re^2]$ since Eq. (3.7) has to be used yields

$$\nabla_\mu u\,\frac{d}{d\tau}D_{\mathbf{1}}{}^{\mu\nu} = 0 \quad . \qquad (4.48)$$

The respective conserved charge

$$p^\nu \equiv \frac{1}{2}\nabla_\mu u\,D_{\mathbf{1}}{}^{\mu\nu} \qquad (4.49)$$

is the full momentum

$$p^\nu = \frac{1}{8\pi}\int d\bar{x}\bar{g}^{1/2}\delta\Big(\Sigma(\bar{x})\Big)g^\nu{}_{\bar{\nu}}(x,\bar{x})\bar{\nabla}_{\bar{\mu}}\Sigma\; J^{\bar{\mu}\bar{\nu}}(\bar{x})\Big|_{x\to\mathcal{I}^+} \quad , \qquad (4.50)$$

and a radiation of the charge is again excluded by the assumption about the support of $J^{\mu\nu}$.

The vector and cross vertices at $\mathcal{I}^+$.

In Eq. (4.29) the moments $D_{\mathbf{1}}$ always appear differentiated with respect to time. This is no more the case in Eqs. (4.30) and (4.31) but in the specific combination of $F(m,n)$ entering the vertex functions (2.41) and (2.42) the terms with the undifferentiated $D_{\mathbf{1}}$

cancel. Using integration by parts we obtain

$$\hat{V}^{\alpha\beta}_{\text{cross}}\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} = -\frac{1}{6r^2}\int\limits_{-\infty}^{u} d\tau\Big\{(\tau-u)^3\Big[\frac{d^2}{d\tau^2}\,D_{\mathbf{1}}{}^{\alpha\beta}(\tau,\phi)\Big]\Big[\frac{d^2}{d\tau^2}\,\hat{D}_{\mathbf{1}}{}^{Q}(\tau,\phi)\Big] \tag{4.51}$$

$$+2(\tau-u)^2\Big[\frac{d^2}{d\tau^2}\,D_{\mathbf{1}}{}^{\alpha\beta}(\tau,\phi)\Big]\Big[\frac{d}{d\tau}\,\hat{D}_{\mathbf{1}}{}^{Q}(\tau,\phi)\Big]\Big\} \quad ,$$

$$\hat{V}^{\alpha\beta}_{\text{vect}}\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} = \frac{1}{12r}\int\limits_{-\infty}^{u} d\tau(\tau-u)^2\Big[\frac{d}{d\tau}\,\hat{D}_{\mathbf{1}}{}^{\alpha}(\tau,\phi)\Big]\Big[\frac{d}{d\tau}\,\hat{D}_{\mathbf{1}}{}^{\beta}(\tau,\phi)\Big] \quad . \tag{4.52}$$

Owing to the presence of the derivatives acting on the $D_{\mathbf{1}}$'s in all terms of these expressions, we have

$$\nabla_\alpha u\,\hat{V}^{\alpha\beta}_{\text{cross}}\Big|_{\mathcal{I}^+} = 0 \quad , \quad \nabla_\alpha u\,\hat{V}^{\alpha\beta}_{\text{vect}}\Big|_{\mathcal{I}^+} = 0 \tag{4.53}$$

by (4.42) and (4.48).

For a tensor $V^{\alpha\beta}$ with the property (4.53) we are to calculate the behaviour at $\mathcal{I}^+[u,\phi,r\to\infty]$ of the quantity $\nabla_\alpha\nabla_\beta V^{\alpha\beta}$ appearing in (2.39). For that, we expand $V^{\alpha\beta}$ over the null-tetrad basis in (1.19):

$$V^{\alpha\beta} = e^\alpha(\mu)e^\beta(\nu)V(\mu\nu) \tag{4.54}$$

and use that

$$\nabla_\alpha u\; e^\alpha(\mu)V(\mu\nu)\Big|_{\mathcal{I}^+} = 0 \tag{4.55}$$

by (4.53), and

$$\nabla e^\alpha(\mu)\Big|_{\mathcal{I}^+} = O\Big(\frac{1}{r}\Big) \quad , \quad \nabla\nabla e^\alpha(\mu)\Big|_{\mathcal{I}^+} = O\Big(\frac{1}{r^2}\Big) \quad , \quad \text{etc.} \tag{4.56}$$

by the result quoted in [23].

Since $V(\mu\nu)$ is a collection of scalars, one has

$$\nabla_\alpha V(\mu\nu) = \Big(\nabla_\alpha u\,\frac{\partial}{\partial u} + \nabla_\alpha r\,\frac{\partial}{\partial r} + \nabla_\alpha\phi^a\frac{\partial}{\partial\phi^a}\Big)V(\mu\nu) \tag{4.57}$$

whence, in view of (4.9),

$$\nabla_\alpha V(\mu\nu)\Big|_{\mathcal{I}^+} = \nabla_\alpha u\,\frac{\partial}{\partial u}V(\mu\nu) + O\Big(\frac{1}{r}V\Big) \quad . \tag{4.58}$$

Similarly,

$$\begin{aligned}\nabla_\alpha\nabla_\beta V(\mu\nu)\Big|_{\mathcal{I}^+} &= \nabla_\alpha u\nabla_\beta u\frac{\partial^2}{\partial u^2}V(\mu\nu) + (\nabla_\alpha u\nabla_\beta r + \nabla_\beta u\nabla_\alpha r)\frac{\partial^2}{\partial u\partial r}V(\mu\nu)\\ &+ (\nabla_\alpha u\nabla_\beta\phi^a + \nabla_\beta u\nabla_\alpha\phi^a)\frac{\partial^2}{\partial u\partial\phi^a}V(\mu\nu) + (\nabla_\alpha\nabla_\beta u)\frac{\partial}{\partial u}V(\mu\nu)\\ &+ O\Big(\frac{1}{r^2}V\Big) \quad .\end{aligned} \tag{4.59}$$

Upon the insertion of these expressions in $\nabla_\alpha \nabla_\beta V^{\alpha\beta}$ and the use of (4.54)-(4.56) along with the condition $\nabla(e^\alpha(\mu)\nabla_\alpha u) = 0$, one is left with

$$\nabla_\alpha \nabla_\beta V^{\alpha\beta}\Big|_{\mathcal{I}^+} = -(\nabla_\alpha \nabla_\beta u)\frac{\partial}{\partial u}V^{\alpha\beta} + O\Big(\frac{1}{r^2}V\Big) \quad . \tag{4.60}$$

An explicit calculation in the metric (1.15) yields

$$\nabla_\alpha \nabla_\beta u\Big|_{\mathcal{I}^+} = -\frac{1}{2r}(m_\alpha m^*_\beta + m^*_\alpha m_\beta) \tag{4.61}$$

whence finally

$$\nabla_\alpha \nabla_\beta V^{\alpha\beta}\Big|_{\mathcal{I}^+} = \frac{1}{r}\frac{\partial}{\partial u}\Big(m_\alpha m^*_\beta V^{\alpha\beta}\Big) + O\Big(\frac{1}{r^2}V\Big) \quad . \tag{4.62}$$

By using (1.21) and (4.53), this result can also be written in the form

$$\nabla_\alpha \nabla_\beta V^{\alpha\beta}\Big|_{\mathcal{I}^+} = \frac{1}{r}\frac{\partial}{\partial u}\Big(g_{\alpha\beta} V^{\alpha\beta}\Big) + O\Big(\frac{1}{r^2}V\Big) \quad . \tag{4.63}$$

One power of $1/r$ is thus gained.

For $\hat{V}^{\alpha\beta}_{\text{cross}}$ in (4.51) this gain is sufficient to satisfy the criterion (4.26):

$$\begin{aligned}\nabla_\alpha \nabla_\beta \hat{V}^{\alpha\beta}_{\text{cross}}\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} \quad = \quad & -\frac{1}{6r^3}\int\limits_{-\infty}^{u} d\tau\Big\{3(u-\tau)^2\Big[\frac{d^2}{d\tau^2}\, D_{\mathbf{1}}{}^R(\tau,\phi)\Big]\Big[\frac{d^2}{d\tau^2}\, \hat{D}_{\mathbf{1}}{}^Q(\tau,\phi)\Big] \\ & -4(u-\tau)\Big[\frac{d^2}{d\tau^2}\, D_{\mathbf{1}}{}^R(\tau,\phi)\Big]\Big[\frac{d}{d\tau}\, \hat{D}_{\mathbf{1}}{}^Q(\tau,\phi)\Big]\Big\}\end{aligned} \tag{4.64}$$

but in the case of $\hat{V}^{\alpha\beta}_{\text{vect}}$ in (4.52) we wind up with the behaviour $1/r^2$:

$$\nabla_\alpha \nabla_\beta \hat{V}^{\alpha\beta}_{\text{vect}}\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} = \frac{1}{6r^2}\int\limits_{-\infty}^{u} d\tau(u-\tau)g_{\alpha\beta}\Big[\frac{d}{d\tau}\, \hat{D}_{\mathbf{1}}{}^\alpha(\tau,\phi)\Big]\Big[\frac{d}{d\tau}\, \hat{D}_{\mathbf{1}}{}^\beta(\tau,\phi)\Big] + O\Big(\frac{1}{r^3}\Big)\,. \tag{4.65}$$

The integrand in the latter expression

$$g_{\alpha\beta}\Big(\frac{d}{d\tau}\hat{D}_{\mathbf{1}}{}^\alpha\Big)\Big(\frac{d}{d\tau}\hat{D}_{\mathbf{1}}{}^\beta\Big) = \Big|\frac{d}{d\tau}D_{\mathbf{1}}{}^\alpha m_\alpha\Big|^2 \tag{4.66}$$

is none other than the energy flux of the outgoing waves of the vector connection field (see Sec.8). To ensure the fulfilment of the criterion (4.26) we are compelled to impose a limitation on the vector source, namely that this source does not radiate classically; then the quantity (4.66) vanishes. The energy of the vacuum radiation is obtained in the present paper under this limitation, and the limitation itself is discussed in conclusion.

The gravitational vertex at $\mathcal{I}^+$.

Using expression (4.31) for the $F(m,n)$ entering the vertex function (2.40), and integrating by parts, we obtain

$$\hat{V}^{\alpha\beta\mu\nu}_{\rm grav}\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} = -\frac{\hat{1}}{360r}\int\limits_{-\infty}^{u} d\tau\Big\{(\tau-u)^4\Big[\frac{d^2}{d\tau^2}{D_{\mathbf{1}}}^{\alpha\beta}(\tau,\phi)\Big]\Big[\frac{d^2}{d\tau^2}{D_{\mathbf{1}}}^{\mu\nu}(\tau,\phi)\Big] \tag{4.67}$$
$$+(\tau-u)^2\Big[\frac{d}{d\tau}{D_{\mathbf{1}}}^{\alpha\beta}(\tau,\phi)\Big]\Big[\frac{d}{d\tau}{D_{\mathbf{1}}}^{\mu\nu}(\tau,\phi)\Big]+2{D_{\mathbf{1}}}^{\alpha\beta}(\tau,\phi){D_{\mathbf{1}}}^{\mu\nu}(\tau,\phi)\Big\}\ .$$

Here, as distinct from the previous case, the terms with the undifferentiated $D_{\mathbf{1}}$ do not cancel. For the completely symmetrized $\hat{V}^{\alpha\beta\mu\nu}_{\rm grav}$ we have

$$\nabla_\alpha u\,\hat{V}^{(\alpha\beta\mu\nu)}_{\rm grav}\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} = -\frac{\hat{1}}{90r}p^{(\beta}\int\limits_{-\infty}^{u} d\tau{D_{\mathbf{1}}}^{\mu\nu)}(\tau,\phi) \tag{4.68}$$

by (4.48) and (4.49). Two more time derivatives and two more contractions with ∇u are needed for this quantity to vanish:

$$\nabla_\alpha u\,\frac{\partial^2}{\partial u^2}\hat{V}^{(\alpha\beta\mu\nu)}_{\rm grav}\Big|_{\mathcal{I}^+} = -\frac{\hat{1}}{270r}\Big\{p^\beta\frac{\partial}{\partial u}{D_{\mathbf{1}}}^{\mu\nu}(u,\phi)+p^\mu\frac{\partial}{\partial u}{D_{\mathbf{1}}}^{\nu\beta}(u,\phi)+p^\nu\frac{\partial}{\partial u}{D_{\mathbf{1}}}^{\beta\mu}(u,\phi)\Big\}\ , \tag{4.69}$$

$$\nabla_\mu u\nabla_\beta u\nabla_\alpha u\,\frac{\partial^2}{\partial u^2}\hat{V}^{(\alpha\beta\mu\nu)}_{\rm grav}\Big|_{\mathcal{I}^+} = 0\quad . \tag{4.70}$$

Above, the tensor $\hat{V}^{\alpha\beta\mu\nu}_{\rm grav}$ is considered symmetrized because, with the appropriate accuracy, it is symmetrized in the quantity appearing in (2.39):

$$\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu\,\hat{V}^{\alpha\beta\mu\nu}_{\rm grav} = \nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu\,\hat{V}^{(\alpha\beta\mu\nu)}_{\rm grav}+O[\Re^3]\quad . \tag{4.71}$$

For the calculation of this quantity at $\mathcal{I}^+$, we introduce the projection of $\hat{V}^{(\alpha\beta\mu\nu)}_{\rm grav}$ on the null tetrad

$$\hat{V}^{(\alpha\beta\mu\nu)}_{\rm grav} = e^\alpha(\gamma)e^\beta(\sigma)e^\mu(\rho)e^\nu(\delta)\,V(\gamma\sigma\rho\delta) \tag{4.72}$$

and use that by (4.70)

$$\nabla_\mu u\nabla_\beta u\nabla_\alpha u\;e^\alpha(\gamma)e^\beta(\sigma)e^\mu(\rho)\frac{\partial^2}{\partial u^2}V(\gamma\sigma\rho\delta)\Big|_{\mathcal{I}^+} = 0\quad . \tag{4.73}$$

One needs only the third and fourth derivatives of $V(\gamma\sigma\rho\delta)$ since by (4.56)

$$
\begin{aligned}
\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu \hat{V}^{(\alpha\beta\mu\nu)}_{\rm grav}\Big|_{\mathcal{I}^+} &= 4\nabla_\alpha\Big(e^\alpha(\gamma)e^\beta(\sigma)e^\mu(\rho)e^\nu(\delta)\Big)\nabla_\beta\nabla_\mu\nabla_\nu\, V(\gamma\sigma\rho\delta) \\
&+ e^\alpha(\gamma)e^\beta(\sigma)e^\mu(\rho)e^\nu(\delta)\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu\, V(\gamma\sigma\rho\delta) + O\Big(\frac{1}{r^2}V\Big) \quad .
\end{aligned} \tag{4.74}
$$

The third derivative is needed only to lowest order,

$$
\nabla_\beta\nabla_\mu\nabla_\nu\, V(\gamma\sigma\rho\delta)\Big|_{\mathcal{I}^+} = \nabla_\beta u\nabla_\mu u\nabla_\nu u\frac{\partial^3}{\partial u^3}V(\gamma\sigma\rho\delta) + O\Big(\frac{1}{r}V\Big) \tag{4.75}
$$

whereas of the fourth derivative one needs two orders:

$$
\begin{aligned}
\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu\, V(\gamma\sigma\rho\delta)\Big|_{\mathcal{I}^+} &= \nabla_\alpha u\nabla_\beta u\nabla_\mu u\nabla_\nu u\frac{\partial^4}{\partial u^4}V(\gamma\sigma\rho\delta) \\
&+ \Big(\nabla_\alpha u\nabla_\beta u\nabla_\mu u\nabla_\nu r + \nabla_\alpha u\nabla_\beta u\nabla_\nu u\nabla_\mu r \\
&\quad +\nabla_\alpha u\nabla_\mu u\nabla_\nu u\nabla_\beta r + \nabla_\beta u\nabla_\mu u\nabla_\nu u\nabla_\alpha r\Big)\frac{\partial^4}{\partial u^3\partial r}V(\gamma\sigma\rho\delta) \\
&+ \Big(\nabla_\alpha u\nabla_\beta u\nabla_\mu u\nabla_\nu \phi^a + \nabla_\alpha u\nabla_\beta u\nabla_\nu u\nabla_\mu \phi^a \\
&\quad +\nabla_\alpha u\nabla_\mu u\nabla_\nu u\nabla_\beta \phi^a + \nabla_\beta u\nabla_\mu u\nabla_\nu u\nabla_\alpha \phi^a\Big)\frac{\partial^4}{\partial u^3\partial\phi^a}V(\gamma\sigma\rho\delta) \\
&+ \Big(\nabla_\mu u\nabla_\nu u\cdot\nabla_\alpha\nabla_\beta u + \nabla_\mu u\nabla_\alpha u\cdot\nabla_\nu\nabla_\beta u \\
&\quad +\nabla_\mu u\nabla_\beta u\cdot\nabla_\nu\nabla_\alpha u + \nabla_\nu u\nabla_\alpha u\cdot\nabla_\mu\nabla_\beta u \\
&\quad +\nabla_\nu u\nabla_\beta u\cdot\nabla_\mu\nabla_\alpha u + \nabla_\alpha u\nabla_\beta u\cdot\nabla_\mu\nabla_\nu u\Big)\frac{\partial^3}{\partial u^3}V(\gamma\sigma\rho\delta) \\
&+ O\Big(\frac{1}{r^2}V\Big) \quad .
\end{aligned} \tag{4.76}
$$

All terms in the two latter expressions have a sufficient number of $\partial/\partial u$ for Eq. (4.73) to work but not all have a sufficient number of ∇u. As a result, one is left with

$$
\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu \hat{V}^{\alpha\beta\mu\nu}_{\rm grav}\Big|_{\mathcal{I}^+} = -6\nabla_\alpha u\nabla_\beta u(\nabla_\mu\nabla_\nu u)\frac{\partial^3}{\partial u^3}\hat{V}^{(\alpha\beta\mu\nu)}_{\rm grav} + O\Big(\frac{1}{r^2}V\Big) \quad . \tag{4.77}
$$

It is already seen that the present case is dissimilar to the previous one (cf. Eq. (4.60)). Since $\nabla\nabla u = O(1/r)$, there can be and is only one factor of $\nabla\nabla u$ in (4.77). The remaining indices have to be contracted with ∇u. Therefore, the completely transverse projection of $\hat{V}^{\alpha\beta\mu\nu}_{\rm grav}$ drops out of (4.77).

Using (4.61) we have

$$
\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu \hat{V}^{\alpha\beta\mu\nu}_{\rm grav}\Big|_{\mathcal{I}^+} = \frac{6}{r}\frac{\partial^3}{\partial u^3}\Big(\nabla_\alpha u\nabla_\beta u\; m_\mu m^*_\nu \hat{V}^{(\alpha\beta\mu\nu)}_{\rm grav}\Big) + O\Big(\frac{1}{r^2}V\Big) \quad , \tag{4.78}
$$

or, by (1.21) and (4.70),

$$\nabla_\alpha \nabla_\beta \nabla_\mu \nabla_\nu \hat{V}^{\alpha\beta\mu\nu}_{\rm grav}\Big|_{\mathcal{I}^+} = \frac{6}{r}\frac{\partial^3}{\partial u^3}\Big(\nabla_\alpha u \nabla_\beta u \; g_{\mu\nu}\hat{V}^{(\alpha\beta\mu\nu)}_{\rm grav}\Big) + O\Big(\frac{1}{r^2}V\Big) \quad . \tag{4.79}$$

Finally, using (4.69),(4.37), and denoting

$$\mu = \nabla_\nu u \, p^\nu \quad , \tag{4.80}$$

we obtain

$$\nabla_\alpha \nabla_\beta \nabla_\mu \nabla_\nu \hat{V}^{\alpha\beta\mu\nu}_{\rm grav}\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} = \frac{\hat{1}}{r^2}\,\frac{\mu}{45}\,\frac{d^2}{du^2}\,D_{\mathbf{1}}{}^R(u,\phi) + O\Big(\frac{1}{r^3}\Big) \quad . \tag{4.81}$$

The result is that the $1/r^2$ term survives also here but there is nothing like a radiation flux in its coefficient. The most important fact is that this coefficient vanishes at late time as distinct from the coefficient in (4.65) which grows at late time (Eqs. (6.53) and (6.54) below). Since only the late-time behaviour is relevant to the calculation of the total vacuum energy, the $1/r^2$ term in (4.81) will have no effect on this calculation and no special limitation on the tensor source will be required.

The convergence condition (4.26) should, of course, be fulfilled everywhere at $\mathcal{I}^+$. The full solution of this problem including a removal of the limitation on the vector source remains beyond the scope of the present work (see the conclusion).

The trees at $\mathcal{I}^+$.

There remain to be considered the terms in (2.38)-(2.39) whose operator coefficients factorize into a product of massless Green functions. The simplest such terms are $\hat{I}_1$ in (2.38) and the trees in $\hat{I}_2$ that contain a curvature at the observation point :

$$R \quad , \quad J^{\alpha\beta}\Big(\frac{1}{\Box}J_{\alpha\beta}\Big) \quad , \quad R\Big(\frac{1}{\Box}R\Big) \quad . \tag{4.82}$$

Generally, in an asymptotically flat spacetime one has (see, e.g., [23])

$$J^{\alpha\beta}\Big|_{\mathcal{I}^+} = O\Big(\frac{1}{r^2}\Big) \quad , \quad R\Big|_{\mathcal{I}^+} = O\Big(\frac{1}{r^3}\Big) \quad , \tag{4.83}$$

and the power of decrease of $(1/\Box)X$ at $\mathcal{I}^+$ is $1/r$. Therefore, in the general case all the trees in (4.82) are $O(1/r^3)$, and with the present assumption about the support of the sources they vanish identically outside a spacetime tube.

The remaining tree in (2.45) can be written in the form

$$\Big(\nabla_\beta \frac{1}{\Box} J^{\alpha\lambda}\Big)\Big(\nabla_\alpha \frac{1}{\Box} J^{\beta}{}_{\lambda}\Big) = \nabla_\alpha \nabla_\beta A^{\alpha\beta} + O[\Re^3] \tag{4.84}$$

where

$$A^{\alpha\beta} = \Big(\frac{1}{\Box} J^{\alpha\lambda}\Big)\Big(\frac{1}{\Box} J^{\beta}{}_{\lambda}\Big) \tag{4.85}$$

and use is made of the conservation law (1.9). By (4.17)

$$A^{\alpha\beta}\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} = \frac{1}{r^2} g_{\mu\nu} D_{\mathbf{1}}{}^{\alpha\mu}(u,\phi) D_{\mathbf{1}}{}^{\beta\nu}(u,\phi) \quad , \tag{4.86}$$

and, to lowest order in $1/r$,

$$\nabla_\alpha \nabla_\beta A^{\alpha\beta}\Big|_{\mathcal{I}^+} = \nabla_\alpha u \nabla_\beta u \frac{\partial^2}{\partial u^2} A^{\alpha\beta} + O\Big(\frac{1}{r} A\Big) \quad . \tag{4.87}$$

Hence

$$\nabla_\alpha \nabla_\beta A^{\alpha\beta}\Big|_{\mathcal{I}^+} = \frac{4}{r^2} \frac{d^2}{du^2} g_{\mu\nu} p^\mu p^\nu + O\Big(\frac{1}{r^3}\Big) = O\Big(\frac{1}{r^3}\Big) \tag{4.88}$$

where p^μ is the conserved momentum (4.49). Thus, also this tree is $O(1/r^3)$ at $\mathcal{I}^+$.

There is, however, one more tree which thus far has not been taken into account. The point is that we are using everywhere the lowest-order approximation (3.1) for the resolvent. This is correct in the terms of second order in $\Re$ but in the terms of first order in $\Re$ the resolvent itself should be taken with a higher accuracy. Therefore, we must come back to expression (2.36) with the first-order I , $I = I_1$, and calculate the curvature correction to the form factor [14] in this expression:

$$\delta \log(-\Box) I_1\Big|_{\mathcal{I}^+} \quad . \tag{4.89}$$

By using the spectral form

$$\log(-\Box/c^2) = -\int_0^\infty dm^2 \Big(\frac{1}{m^2 - \Box} - \frac{1}{m^2 + c^2}\Big) \tag{4.90}$$

[14]The curvature corrections to the derivatives $\nabla^\mu\nabla^\nu$ in (2.36) vanish at $\mathcal{I}^+$.

and the variational law for the retarded Green function we obtain

$$\delta\log(-\Box)X(x) = -\int\limits_0^\infty dm^2 \frac{1}{m^2-\Box_3}\delta\Box_2\frac{1}{m^2-\Box_1}X(x) = \frac{\log(\Box_3/\Box_1)}{\Box_3-\Box_1}\delta\Box_2 X(x) \qquad (4.91)$$

where the numbers on the $\Box$'s indicate the order in which the operators act on $X(x)$. We may now use the theorem in [21,22] by which the limit of (4.91) as $x\to\mathcal{I}^+$ is determined by the limit $\Box\to 0$ in the operator $\Box$ acting the last i.e. in $\Box_3$. We obtain

$$\delta\log(-\Box)X\Big|_{\mathcal{I}^+} = -\frac{\log(-\Box_3)}{\Box_1}\delta\Box_2 X = -\log(-\Box)\delta\Box\frac{1}{\Box}X \quad . \qquad (4.92)$$

As could be expected, the variation of $\log(-\Box)$ in the background fields behaves at $\mathcal{I}^+$ in the same way as $\log(-\Box)$ itself, i.e. like $1/r^2$, and the result (4.92) is valid up to $O(1/r^3)$.

Using the latter result in expression (4.89), we may write

$$\Big(\log(-\Box)+\delta\log(-\Box)\Big)I_1\Big|_{\mathcal{I}^+} = \log(-\Box)\Big(I_1 - \delta\Box\frac{1}{\Box}I_1\Big) \quad . \qquad (4.93)$$

This means that introducing a correction to the form factor boils down to the replacement of the scalar I by $I+\delta I$ with

$$\delta I = -\delta\Box\frac{1}{\Box}I_1 \quad . \qquad (4.94)$$

Upon this replacement one may use everywhere the lowest-order approximation for the resolvent. In (4.94) one is to insert the expression for I_1 from Eq. (2.38) and the expression

$$\delta\Box = 2\Big(\frac{1}{\Box}R^{\alpha\beta}\Big)\nabla_\alpha\nabla_\beta \qquad (4.95)$$

from Ref. [15]. As a result, the correction under the sign of trace in (2.37) takes the form

$$\delta\hat{I} = \frac{\hat{1}}{90}\Big(\frac{1}{\Box}R^{\alpha\beta}\Big)\nabla_\alpha\nabla_\beta\frac{1}{\Box}R \quad , \qquad (4.96)$$

or, in terms of the conserved current $J^{\alpha\beta}$,

$$\delta\hat{I} = \frac{\hat{1}}{90}\nabla_\alpha\nabla_\beta B^{\alpha\beta} + \frac{\hat{1}}{180}R\Big(\frac{1}{\Box}R\Big) \qquad (4.97)$$

with

$$B^{\alpha\beta} = \Big(\frac{1}{\Box}J^{\alpha\beta}\Big)\Big(\frac{1}{\Box}R\Big) \quad . \qquad (4.98)$$

Expression (4.97) is one more tree.

The second term in (4.97) is of the type (4.82), and its contribution at $\mathcal{I}^+$ is $O(1/r^3)$. The calculation of the first term at $\mathcal{I}^+$ is similar to the one in Eqs. (4.86)-(4.87):

$$B^{\alpha\beta}\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} = \frac{1}{r^2} D_{\mathbf{1}}{}^{\alpha\beta}(u,\phi) D_{\mathbf{1}}{}^{R}(u,\phi) \quad , \tag{4.99}$$

$$\nabla_\alpha \nabla_\beta B^{\alpha\beta}\Big|_{\mathcal{I}^+} = \nabla_\alpha u \nabla_\beta u \frac{\partial^2}{\partial u^2} B^{\alpha\beta} + O\Big(\frac{1}{r} B\Big) \tag{4.100}$$

but the result is different. We obtain

$$\delta \hat{I}\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} = \frac{\hat{1}}{r^2} \frac{\mu}{45} \frac{d^2}{du^2} D_{\mathbf{1}}{}^{R}(u,\phi) + O\Big(\frac{1}{r^3}\Big) \tag{4.101}$$

with μ in (4.80). Here the $1/r^2$ term survives but it has the same form as (4.81), and the same inference applies.

5 The early-time behaviours

Let $\mathcal{T}$ be the support tube of the physical sources in (1.8). By the original assumption about the asymptotic stationarity of external fields, in the past and future of tube $\mathcal{T}$ ($\mathcal{T}^-$ and $\mathcal{T}^+$ respectively) there exist asymptotic timelike Killing vectors such that all sources J in (1.8) are conserved along their integral curves. Let $\xi^\alpha(x)$ be a timelike vector field that interpolates between the Killing vectors at $\mathcal{T}^-$ and $\mathcal{T}^+$, and up to $O[\Re]$ is the Killing vector for the whole of $\mathcal{T}$. Denote $\mathcal{L}_\xi$ the Lie derivative in the direction of ξ^α, and

$$\overset{\bullet}{J} = \left(\mathcal{L}_\xi J^{\mu\nu},\ \mathcal{L}_\xi \hat{J}^\mu,\ \mathcal{L}_\xi \hat{P} \right) . \tag{5.1}$$

For simplicity, the support of $\overset{\bullet}{J}$ will be assumed compact. Then, on the central geodesic of any Bondi-Sachs frame, there will be two points o^- and o^+ with $u(o^-) = u^-,\ u(o^+) = u^+$, and $u^- < u^+$, such that the support of $\overset{\bullet}{J}$ is entirely inside the future light cone of o^- and the past light cone of o^+. In the approximation (3.13) the fields of nonstationary sources propagate only between the cones $u = u^-$ and $u = u^+$ rather than in the whole causal future of supp $\overset{\bullet}{J}$. We have

$$\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu = 0 \quad , \quad \overset{\bullet}{J} = 0 \, , \qquad u < u^-,\ \ u > u^+ \quad . \tag{5.2}$$

For what follows we need some properties that the functions involved in the calculation possess in presence of the Killing vector field.

Geometrical two-point functions in presence of the Killing vector.

If ξ^α is a Killing vector field, then the Lie derivative $\mathcal{L}_\xi$ commutes with the covariant derivative ∇^β when acting on any object:

$$\mathcal{L}_\xi \nabla^\beta X^{\alpha\dots} = \nabla^\beta \mathcal{L}_\xi X^{\alpha\dots} \quad . \tag{5.3}$$

Indeed, by definition,

$$\mathcal{L}_\xi X^{\alpha\dots} = \xi^\mu \nabla_\mu X^{\alpha\dots} - \Sigma X^{\mu\dots} \nabla_\mu \xi^\alpha \tag{5.4}$$

whence making use of the Killing equation one obtains

$$\mathcal{L}_\xi \nabla^\beta X^{\alpha\dots} - \nabla^\beta \mathcal{L}_\xi X^{\alpha\dots} = \Sigma X^{\mu\dots}\Big(\nabla^\beta \nabla_\mu \xi^\alpha - R^{\alpha\cdot\beta}_{\cdot\mu\cdot\nu}\xi^\nu\Big) \quad , \tag{5.5}$$

and this commutator vanishes by the known property of the Killing vector field [25]

$$\nabla_\alpha \nabla_\beta \, \xi_\gamma = R_{\gamma\beta\alpha\cdot}{}^{\sigma}\xi_\sigma \quad . \tag{5.6}$$

Consider now the world function $\sigma = \sigma(x, \bar{x})$ for a timelike or spacelike separation of the points x and $\bar{x}$. The vector $n = \nabla\sigma/\sqrt{2|\sigma|}$ is a unit tangent at the point x to the geodesic connecting x and $\bar{x}$, and the vector $\bar{n} = \bar{\nabla}\sigma/\sqrt{2|\sigma|}$ is the oppositely directed unit tangent to the same geodesic at the point $\bar{x}$. Since the quantity $\xi^\alpha n_\alpha$ is conserved along the geodesic [25], we have

$$\xi^\alpha n_\alpha = -\bar{\xi}^\alpha \bar{n}_\alpha \quad , \tag{5.7}$$

or

$$\xi^\alpha \nabla_\alpha \sigma(x, \bar{x}) + \bar{\xi}^\alpha \bar{\nabla}_\alpha \sigma(x, \bar{x}) = 0 \quad . \tag{5.8}$$

This is the conservation law for the world function. By continuity it holds also for a null separation of the points x and $\bar{x}$.

Eq. (5.8) can be written in the form

$$\Big(\mathcal{L}_\xi + \mathcal{L}_{\bar{\xi}}\Big)\sigma = 0 \tag{5.9}$$

whence using (5.3) one obtains

$$\Big(\mathcal{L}_\xi + \mathcal{L}_{\bar{\xi}}\Big)\nabla_{\alpha_1}\dots\nabla_{\alpha_n}\bar{\nabla}_{\beta_1}\dots\bar{\nabla}_{\beta_m}\sigma = 0 \tag{5.10}$$

— a conservation law for the derivatives of the world function.

A similar law for the propagator of the geodetic parallel transport can be obtained as follows. From the equations (3.3) for $g^\alpha{}_{\bar{\alpha}}$ we have

$$\Big(\mathcal{L}_\xi \sigma^\mu\Big)\nabla_\mu g^{\alpha\bar{\alpha}} + \sigma^\mu \nabla_\mu \Big(\mathcal{L}_\xi g^{\alpha\bar{\alpha}}\Big) \;\; = \;\; 0 \quad , \tag{5.11}$$

$$\Big(\mathcal{L}_{\bar{\xi}} \sigma^\mu\Big)\nabla_\mu g^{\alpha\bar{\alpha}} + \sigma^\mu \nabla_\mu \Big(\mathcal{L}_{\bar{\xi}} g^{\alpha\bar{\alpha}}\Big) \;\; = \;\; 0 \tag{5.12}$$

where use is made of the commutation law (5.3). Since

$$\Big(\mathcal{L}_{\xi}+\mathcal{L}_{\bar\xi}\Big)\sigma^{\mu}=0 \tag{5.13}$$

by (5.10), combining Eqs. (5.11) and (5.12) yields

$$\sigma^{\mu}\nabla_{\mu}\Big(\mathcal{L}_{\xi}+\mathcal{L}_{\bar\xi}\Big)g^{\alpha\bar\alpha}=0 \quad . \tag{5.14}$$

We have also

$$\Big(\mathcal{L}_{\xi}+\mathcal{L}_{\bar\xi}\Big)g^{\alpha\bar\alpha}\Big|_{\bar x=x}=-\nabla^{\bar\alpha}\xi^{\alpha}-\nabla^{\alpha}\xi^{\bar\alpha}=0 \tag{5.15}$$

which follows from the initial condition in (3.3) and the condition

$$\nabla_{\mu}g^{\alpha\bar\alpha}\Big|_{\bar x=x}=0 \quad . \tag{5.16}$$

The latter condition is obtained by differentiating the equation for $g^{\alpha}{}_{\bar\alpha}$ in (3.3) and setting $\bar x=x$. With the initial condition (5.15), the solution of Eq. (5.14) is

$$\Big(\mathcal{L}_{\xi}+\mathcal{L}_{\bar\xi}\Big)g^{\alpha\bar\alpha}=0. \tag{5.17}$$

This is the desired result.

Finally, consider Eq. (5.8) with a timelike ξ^{α} and go over to the limit $x\to\mathcal{I}^{+}$ in this equation. The insertion of the asymptotic behaviour of the world function (4.5) yields

$$-\xi^{\mu}\nabla_{\mu}u(x)+\xi^{\mu}\nabla_{\mu}\phi^{a}(x)\frac{\partial}{\partial\phi^{a}}U_{\phi}(\bar x)=-\bar\xi^{\mu}\bar\nabla_{\mu}U_{\phi}(\bar x) \quad , \quad x\to\mathcal{I}^{+} \quad . \tag{5.18}$$

Here the term with $\partial/\partial\phi$ drops out by (4.9), and, with the usual normalization of the timelike Killing vector at infinity, we have

$$\xi^{\mu}\nabla_{\mu}u\Big|_{\mathcal{I}^{+}}=1 \quad . \tag{5.19}$$

Thus we obtain

$$\bar\xi^{\mu}\bar\nabla_{\mu}U_{\phi}(\bar x)=1 \tag{5.20}$$

which is the conservation law for the null hyperplanes. Using (5.3) we obtain also

$$\mathcal{L}_{\bar\xi}\bar\nabla_{\alpha_1}\dots\bar\nabla_{\alpha_n}U_{\phi}(\bar x)=0 \quad , \quad n\geq 1 \quad . \tag{5.21}$$

In the Bondi-Sachs frame with the retarded time normalized as in (1.14), the solution of Eq. (5.20) is

$$U_\phi(\bar{u},\bar{\phi},\bar{r}) = \bar{u} + L_\phi(\bar{\phi},\bar{r}) \quad , \tag{5.22}$$

where the function $L_\phi(\bar{\phi},\bar{r})$ possesses the properties

$$L_\phi(\bar{\phi},\bar{r}) \geq 0 \quad , \quad L_\phi(\bar{\phi},\bar{r})\Big|_{\bar{\phi}=\phi} = 0 \quad , \quad \frac{\partial}{\partial\bar{\phi}} L_\phi(\bar{\phi},\bar{r})\Big|_{\bar{\phi}=\phi} = 0 \tag{5.23}$$

by (4.13) and (4.14).

It should be emphasized that the relations above for the two-point functions hold only in the case where the geodesic connecting the two points lies entirely in the Killing domain.

Retarded kernels in presence of the Killing vector.

That an arbitrary timelike vector in Eq. (3.12) and similar equations has been denoted ξ^α is no mere coincidence. As will be seen from the calculations below, it is advantageous to choose for this arbitrary vector the vector ξ^α defined in the beginning of the present section.

Consider an integral of some scalar source X over the past hyperboloid of a point x, and calculate

$$\begin{aligned}
&\xi^\alpha \nabla_\alpha \int\limits_{\text{past of } x} d\bar{x}\bar{g}^{1/2} \delta\Big(\sigma(x,\bar{x}) - q\Big)\bar{X} \qquad\qquad (5.24)\\
&\qquad = \int\limits_{\text{past of } x} d\bar{x}\bar{g}^{1/2} \frac{(\xi\cdot\nabla\sigma)}{(\bar{\xi}\cdot\bar{\nabla}\sigma)} \bar{\xi}^\alpha \bar{\nabla}_\alpha \delta\Big(\sigma(x,\bar{x}) - q\Big)\bar{X}\\
&\qquad = -\int\limits_{\text{past of } x} d\bar{x}\bar{g}^{1/2} \delta\Big(\sigma(x,\bar{x}) - q\Big)\bar{\nabla}_\alpha\Big(\bar{\xi}^\alpha \frac{(\xi\cdot\nabla\sigma)}{(\bar{\xi}\cdot\bar{\nabla}\sigma)}\bar{X}\Big) \quad .
\end{aligned}$$

Let now the observation point x belong to the past Killing domain $u(x) < u^-$. Since the hyperboloid in (5.24) lies entirely in the causal past of x, the integration point $\bar{x}$ also belongs to the past Killing domain. Furthermore, the timelike geodesic connecting x and $\bar{x}$ lies inside the past light cone of x and, therefore, passes entirely through the Killing

domain. Then, using Eq. (5.8) and the corollary $\bar{\nabla}_\alpha \bar{\xi}^\alpha = 0$ of the Killing equation, we obtain

$$\xi^\alpha \nabla_\alpha \int\limits_{\text{past of } x} d\bar{x}\bar{g}^{1/2}\delta\Big(\sigma(x,\bar{x}) - q\Big)\bar{X} = \int\limits_{\text{past of } x} d\bar{x}\bar{g}^{1/2}\delta\Big(\sigma(x,\bar{x}) - q\Big)\bar{\xi}^\alpha\bar{\nabla}_\alpha\bar{X} \quad , \qquad u(x) < u^- \quad . \tag{5.25}$$

For obtaining a similar result in the case of a tensor source X, consider $\bar{X}$ in Eq.(3.8). Write

$$\bar{\xi}^\alpha\bar{\nabla}_\alpha\bar{X} = \mathcal{L}_{\bar{\xi}}\Big(g^{\mu_1}_{\ \bar{\mu}_1}\dots g^{\mu_n}_{\ \bar{\mu}_n}\Big)X^{\bar{\mu}_1\dots\bar{\mu}_n}(\bar{x}) + g^{\mu_1}_{\ \bar{\mu}_1}\dots g^{\mu_n}_{\ \bar{\mu}_n}\mathcal{L}_{\bar{\xi}}X^{\bar{\mu}_1\dots\bar{\mu}_n}(\bar{x}) \tag{5.26}$$

and use the law (5.17). It is then seen that Eq. (5.25) generalizes as follows:

$$\mathcal{L}_\xi \int\limits_{\text{past of } x} d\bar{x}\bar{g}^{1/2}\delta\Big(\sigma(x,\bar{x}) - q\Big)g^{\mu_1}_{\ \bar{\mu}_1}\dots g^{\mu_n}_{\ \bar{\mu}_n}\bar{X}^{\bar{\mu}_1\dots\bar{\mu}_n} = \tag{5.27}$$

$$= \int\limits_{\text{past of } x} d\bar{x}\bar{g}^{1/2}\delta\Big(\sigma(x,\bar{x}) - q\Big)g^{\mu_1}_{\ \bar{\mu}_1}\dots g^{\mu_n}_{\ \bar{\mu}_n}\mathcal{L}_{\bar{\xi}}\bar{X}^{\bar{\mu}_1\dots\bar{\mu}_n} \quad , \quad u(x) < u^- \quad .$$

Eq. (5.27) implies that, in the past Killing domain, the Lie derivative $\mathcal{L}_\xi$ commutes with the operator $\mathcal{H}_q$ in (3.20):

$$\mathcal{L}_\xi\mathcal{H}_q X(x) = \mathcal{H}_q\mathcal{L}_\xi X(x) \quad , \quad u(x) < u^- \quad . \tag{5.28}$$

It then commutes with all the vertex operators:

$$\mathcal{L}_\xi F(m,n)X_1X_2(x) = F(m,n)\Big(\mathcal{L}_\xi X_1\Big)X_2 + F(m,n)X_1\Big(\mathcal{L}_\xi X_2\Big) \quad , \tag{5.29}$$

$$\mathcal{L}_\xi \frac{1}{\Box_2}F(m,n)X_1X_2(x) = \frac{1}{\Box_2}F(m,n)\Big(\mathcal{L}_\xi X_1\Big)X_2 + \frac{1}{\Box_2}F(m,n)X_1\Big(\mathcal{L}_\xi X_2\Big) \quad , \tag{5.30}$$

$$\mathcal{L}_\xi \frac{1}{\Box_1\Box_2}F(m,n)X_1X_2(x) = \frac{1}{\Box_1\Box_2}F(m,n)\Big(\mathcal{L}_\xi X_1\Big)X_2 + \frac{1}{\Box_1\Box_2}F(m,n)X_1\Big(\mathcal{L}_\xi X_2\Big) \quad , \tag{5.31}$$

$$u(x) < u^-$$

and, since (5.28) is valid for $q = 0$ as well, it commutes with the retarded operator $1/\Box$:

$$\mathcal{L}_\xi \frac{1}{\Box}X(x) = \frac{1}{\Box}\mathcal{L}_\xi X(x) \quad , \quad u(x) < u^- \quad . \tag{5.32}$$

Causality of the vacuum radiation.

For any of the physical sources J the commutation relations above yield

$$
\begin{aligned}
\mathcal{L}_\xi \frac{1}{\Box} J(x) &= \frac{1}{\Box} \overset{\bullet}{J}(x) = 0 \quad , & (5.33)\\
\mathcal{L}_\xi \mathcal{H}_q J(x) &= \mathcal{H}_q \overset{\bullet}{J}(x) = 0 \quad , & (5.34)\\
\mathcal{L}_\xi F(m,n) J_1 J_2(x) &= F(m,n) \overset{\bullet}{J}_1 J_2(x) + F(m,n) J_1 \overset{\bullet}{J}_2(x) = 0 \ , & (5.35)\\
& \qquad\qquad u(x) < u^-
\end{aligned}
$$

and similarly with the other vertex operators. Here use is made of Eq. (5.2) and of the retardation of all kernels. The relations above and the commutativity of $\mathcal{L}_\xi$ with the covariant derivative suffice to infer that the scalar $I(x)$ in Eq. (2.37) possesses the property

$$
\xi^\alpha \nabla_\alpha I(x) = 0 \quad , \quad u(x) < u^- \quad . \tag{5.36}
$$

Consider now the moment $D_{\mathbf{1}}$ of the scalar I, and calculate

$$
\begin{aligned}
\frac{\partial}{\partial u} D_{\mathbf{1}}(u,\phi|I) &= -\frac{1}{4\pi}\int d\bar{x}\bar{g}^{1/2}\delta'\Big(U_\phi(\bar{x}) - u\Big)\bar{I} & (5.37)\\
&= \frac{1}{4\pi}\int d\bar{x}\bar{g}^{1/2}\delta\Big(U_\phi(\bar{x}) - u\Big)\bar{\nabla}_\alpha\Big(\frac{\bar{\xi}^\alpha}{(\bar{\xi}\cdot\bar{\nabla}U_\phi)}\bar{I}\Big) \quad .
\end{aligned}
$$

On the integration hyperplane in (5.37) we have $u = U_\phi(\bar{x}) \geq u(\bar{x})$ by (4.13). Therefore, with $u < u^-$ we have $u(\bar{x}) < u^-$ for the whole of the hyperplane. Furthermore, the null geodesic connecting the point $\bar{x}$ of the hyperplane with the point u, ϕ at $\mathcal{I}^+$ belongs to this hyperplane itself. Therefore, we may use Eq. (5.20) and the Killing equation to obtain

$$
\frac{\partial}{\partial u} D_{\mathbf{1}}(u,\phi|I)\Big|_{u<u^-} = \frac{1}{4\pi}\int d\bar{x}\bar{g}^{1/2}\delta\Big(U_\phi(\bar{x}) - u\Big)\bar{\xi}^\alpha\bar{\nabla}_\alpha\bar{I} \quad . \tag{5.38}
$$

Hence, by (5.36),

$$
\frac{\partial}{\partial u} D_{\mathbf{1}}(u,\phi|I)\Big|_{u<u^-} = 0 \quad . \tag{5.39}
$$

Recalling that the flux of the vacuum energy is expressed through the moment in (5.39) by Eq. (4.20), we arrive at the following final result:

$$
\frac{dM(u)}{du}\Big|_{u<u^-} = 0 \quad . \tag{5.40}
$$

The classical radiation is also governed by the time derivatives of the moments $D_{\mathbf{1}}$ but these are the moments of directly the physical sources J. Also for these moments we have

$$\frac{\partial}{\partial u} D_{\mathbf{1}}(u,\phi|J)\Big|_{u<u^-} = 0 \tag{5.41}$$

by (5.2).

We proved the following assertions. Stationary sources radiate neither classically nor quantum-mechanically. Moreover, they don't produce even quantum noise. Radiation, including the uncertain oscillations of the energy flux, starts not earlier than the first light signal from a nonstationary source reaches the observer at infinity.

The proof of these assertions given above is adjusted to the approximations made in the present paper but the assertions themselves are valid beyond these approximations. By using the commutation law (5.3) and the boundary condition of retardation, it is not difficult to show that, in the past Killing domain, the operator $\mathcal{L}_\xi$ commutes with the exact retarded resolvent. Then it commutes with all retarded form factors of the form

$$\Gamma(\Box_1,\dots\Box_n) = \int \frac{dm_1^2\dots dm_n^2 \rho(m_1^2,\dots m_n^2)}{(\Box_1 - m_1^2)\dots(\Box_n - m_n^2)} \quad . \tag{5.42}$$

In other words, if the source and background fields in the equation $(\Box - m^2)\varphi = J$ are static in the past, then so is the retarded solution φ and so are all functions

$$\Gamma(\Box_1,\dots\Box_n)J_1\dots J_n \quad . \tag{5.43}$$

Furthermore, as remarked in Sec.2, expression (2.48) is exact, and I in this expression is generally a sum of terms (5.43). Owing to the presence of the overall time derivatives in (2.48), Eq. (5.40) is an exact fact.

Convergence of the vertex operators.

We may now come back to the question of convergence of the vertex operators in (3.23) and (3.32)-(3.33). The vertex operators act directly on the physical sources J and are expressed through the operator $\mathcal{H}_q$ in (3.20). By rewriting the integration measure in

(3.20) in the form

$$d\bar{x}\bar{g}^{1/2} = \frac{d\sigma d\bar{\Sigma}}{\sqrt{-(\bar{\nabla}\sigma)^2}} \tag{5.44}$$

where $d\bar{\Sigma}$ is the induced volume element on the hyperboloid $\sigma(x,\bar{x}) = q$, and using that on this hyperboloid $(\bar{\nabla}\sigma)^2 = 2\sigma = 2q$, we obtain

$$\mathcal{H}_q J(x) = \frac{1}{4\pi\sqrt{-2q}} \int\limits_{\text{past sheet of } \sigma(x,\bar{x})=q} d\bar{\Sigma}\bar{J} \qquad , \quad q < 0 \quad . \tag{5.45}$$

It is important that , with x fixed and $q \to -\infty$, the entire past hyperboloid $\sigma(x,\bar{x}) = q$ shifts to the past and finds itself in the Killing domain where the sources J are stationary. Since, in addition, the support of J is confined to a spacetime tube whose intersection with the hyperboloid $\sigma(x,\bar{x}) = q$ remains compact as $q \to -\infty$, the integral in (5.45) tends to a finite limit. Therefore, with x fixed and $q \to -\infty$ we have

$$\mathcal{H}_q J(x)\Big|_{q\to-\infty} \propto \frac{1}{\sqrt{-q}} \quad . \tag{5.46}$$

It follows from (5.46) that, with the sources asymptotically static in the past, the vertex functions (3.23) converge for all m and n, and so do the functions (3.32) but the functions (3.33) diverge logarithmically for all m and n. This is a consequence of the singularity at the zero-mass limit of the operator $1/(\Box - m^2)^2$ applied to static sources: the integral (3.26) with such sources diverges at $m^2 = 0$.

The remedy is in the fact that the chain of retarded kernels connecting $T^{\mu\nu}_{\text{vac}}\Big|_{\mathcal{I}^+}$ with the sources J contains time derivatives. This is, in particular, the derivative $\partial/\partial u$ in Eq. (4.18). Up to higher-order terms in $\Re$ it can be commuted with all kernels and considered as acting directly on one of the J 's in the guise of the Lie derivative $\mathcal{L}_\xi$. This is seen from Eqs. (5.38),(5.31), etc. which, up to higher-order terms in $\Re$, hold everywhere by the definition of the vector field ξ^α. When dealing with the vertex function (3.33), this commutation should always be assumed done.

If at least one of the two sources in the vertex function has the support properties of $\overset{\bullet}{J}$, all vertex functions converge including the function (3.33). Indeed, the support of $\overset{\bullet}{J}$ is compact and, therefore, no matter where the point x is located [15] , at the limit

[15] If the point x is located at $u(x) < u^-$, all past hyperboloids of x are outside the support of $\overset{\bullet}{J}$, Eq. (5.34).

$q \to -\infty$ the past hyperboloid of x will go out of this support (see Fig.5). Let q(x) be the parameter q of the earliest hyperboloid of x that still crosses the support of $\overset{\bullet}{J}$. Then Eq. (5.46) gets replaced with

$$\mathcal{H}_q \overset{\bullet}{J}(x)\Big|_{q<\mathrm{q}(x)} = 0 \ , \tag{5.47}$$

and all integrals (3.23),(3.32),(3.33) with one of the sources $X = \overset{\bullet}{J}$ acquire a cut off at the lower limit irrespectively of the nature of the other source.

That the vertex function (3.33) with one of the sources $X = \overset{\bullet}{J}$ converges is seen also from the fact that any power of the massless retarded operator $1/\Box$ applied to $\overset{\bullet}{J}$, $(1/\Box^n)\overset{\bullet}{J}$, is well defined since, by (5.33), all functions $(1/\Box^{n-1})\overset{\bullet}{J}$ vanish identically at $\mathcal{I}^-$. The integrand in (3.26) is then no more singular at $m^2 = 0$ than in the convergent integral (3.25).

The presence of $\overset{\bullet}{J}$ among the sources ensures the convergence of an arbitrary superposition of retarded kernels since any convergent integral of the form

$$Y(x) = \int dx_1 g_1^{1/2} \dots dx_n g_n^{1/2} G(x|x_1, \dots x_n)\, \overset{\bullet}{J}_1\, X_2 \dots X_n \tag{5.48}$$

where G is a retarded kernel possesses the property

$$Y(x) = 0 \ , \quad u(x) < u^- \ . \tag{5.49}$$

Owing to (5.49), the function $Y(x)$ vanishes identically not only at the past timelike and null infinities (i^- and $\mathcal{I}^-$) but also at spatial infinity (i^0):

$$Y(x)\Big|_{x\to i^- \text{ or } \mathcal{I}^- \text{ or } i^0} \equiv 0 \ . \tag{5.50}$$

This is illustrated in Fig.5 whence it is seen that, at any of these limits, the past light cone of x will go out of the support of $\overset{\bullet}{J}$. Owing to the property (5.50), any integral of $Y(x)$ with a retarded kernel converges, and any convergent integral of the form

$$Z(x) = \int dx_1 g_1^{1/2} \dots dx_n g_n^{1/2} G(x|x_1, \dots x_n) Y_1 X_2 \dots X_n \tag{5.51}$$

possesses again the property (5.49):

$$Z(x) = 0 \ , \quad u(x) < u^- \ . \tag{5.52}$$

It is worth emphasizing that the support of a function like the $Y(x)$ or $Z(x)$ above is no more compact and, moreover, is not even a spacetime tube since it has a null boundary. This support (shown with broken lines in Fig.5) is the causal future of the support of $\overset{\bullet}{J}$. Therefore, the presence of time derivatives in the kernels has generally no effect on their behaviours at the future infinities beginning with the future null infinity for $u > u^{-}$. The integrals (4.51),(4.52), etc. are, however, cut from below owing to (5.41).

6 The late-time behaviours

Eq. (5.40) proves that the total energy of vacuum radiation is determined indeed by the limit of late time in (2.49). From (4.20) and (5.39) we have

$$M(-\infty) - M(\infty) = -\lim_{u\to\infty} \frac{2}{(4\pi)^2} \frac{\partial^2}{\partial u^2} \int d^2\mathcal{S}(\phi)\, D_{\mathbf{1}}(u,\phi|I) \tag{6.1}$$

whence it follows that we are to study the behaviour of the moment $D_{\mathbf{1}}$ as $u \to \infty$, i.e. the behaviour of the retarded Green function (4.17) in the future of $\mathcal{I}^+$. For (6.1) to be finite and nonvanishing, this behaviour should be

$$D_{\mathbf{1}}(u,\phi|I)\Big|_{u\to\infty} \propto u^2 \quad . \tag{6.2}$$

The retarded Green function in the future of $\mathcal{I}^+$.

We shall show that the behaviour of

$$D_{\mathbf{1}}(u,\phi|X) \equiv D_{\mathbf{1}}(u,\phi) \tag{6.3}$$

as $u \to \infty$ is determined by competing behaviours of $X(x)$ at the following four limits.

i) The limit of $X(x)$ as $u(x) \to \infty$ along the timelike lines filling a spacetime tube $(\mathcal{T})$. For such lines one can take the lines $r = \text{const.}, \phi = \text{const.}$ of an arbitrarily chosen Bondi-Sachs frame. This limit will be denoted

$$X_{\mathcal{T}^+}[r,\phi,u\to\infty] \quad . \tag{6.4}$$

ii) The limit of $X(x)$ as x moves to the future along the timelike geodesics that reach the asymptotically flat infinity. In the Bondi-Sachs coordinates these geodesics are asymptotically of the form

$$r = \frac{\gamma}{\sqrt{1-\gamma^2}} s \;\;,\;\; u = \frac{\sqrt{1-\gamma}}{\sqrt{1+\gamma}} s \;\;,\;\; \phi = \text{const.} \;\;,\;\; s\to\infty \;\;,\;\; 0<\gamma<1 \tag{6.5}$$

where s is the proper time, and γ is the boost parameter (1.22). This limit has been denoted in Introduction

$$X_{i^+}[\gamma,\phi,s\to\infty] \quad . \tag{6.6}$$

iii) The limit of $X(x)$ as x moves to the future along the null geodesics

$$u = \text{const.} \ , \ \ \phi = \text{const.} \ , \ \ r \to \infty \ . \tag{6.7}$$

This limit has been denoted

$$X_{\mathcal{I}^+}[u, \phi, r \to \infty] \ . \tag{6.8}$$

iiii) The limit of $X(x)$ as x moves to the asymptotically flat infinity along the spacelike geodesics. In the Bondi-Sachs coordinates these geodesics are asymptotically of the form

$$u = -(1-\beta)r \ , \ \ \phi = \text{const.} \ , \ \ r \to \infty \ , \ \ -1 < \beta < 1 \ . \tag{6.9}$$

This limit will be denoted

$$X_{i^0}[\beta, \phi, r \to \infty] \ . \tag{6.10}$$

Consider the integration hyperplane in (4.19). One of its null generators is radial i.e. crosses the central geodesic of the Bondi-Sachs frame at some point o. As follows from the consideration in Sec.4 (and as illustrated by Fig.4), the integration hyperplane lies outside both sheets of the light cone of o. Therefore, the support of $\bar{X}$ in (4.19) is confined to the exterior of this cone. In Fig.6 the light cone of o is depicted with bold lines, and its exterior is divided into four subdomains. The support of $\bar{X}$ in (4.19) is their union

$$\text{supp } \bar{X} = \text{I} \cup \text{II} \cup \text{III} \cup \text{IV} \ , \tag{6.11}$$

and the integral (4.19) itself is a sum of the respective four contributions

$$D_{\mathbf{1}} = D_{\mathbf{1}}^{\text{I}} + D_{\mathbf{1}}^{\text{II}} + D_{\mathbf{1}}^{\text{III}} + D_{\mathbf{1}}^{\text{IV}} \ . \tag{6.12}$$

Subdomain I belongs to a tube $r < r_0$ with r_0 sufficiently large for subdomains $\text{II}, \text{III}, \text{IV}$ to be already in the asymptotically flat zone. Subdomain III is bounded by two future light cones $\bar{u} = u_1$ and $\bar{u} = u_2$ with large positive $u_2 = |u_2|$ and large negative $u_1 = -|u_1|$. With the Bondi-Sachs parametrization of the integrand in (4.19) we have

$$D_{\mathbf{1}}^{\text{I}}(u, \phi) = \frac{1}{4\pi} \int d^2\mathcal{S}(\bar{\phi}) \int\limits_0^{r_0} d\bar{r}\bar{r}^2 \int\limits_{-\infty}^{\infty} d\bar{u} \Big|(\bar{\nabla}\bar{u}, \bar{\nabla}\bar{r})\Big|^{-1} \delta\Big(U_\phi(\bar{u}, \bar{\phi}, \bar{r}) - u\Big)\bar{X} \ , \tag{6.13}$$

$$D_{\mathbf{1}}^{\text{II}}(u, \phi) = \frac{1}{4\pi} \int d^2\mathcal{S}(\bar{\phi}) \int\limits_{r_0}^{\infty} d\bar{r}\bar{r}^2 \int\limits_{u_2}^{\infty} d\bar{u} \Big|(\bar{\nabla}\bar{u}, \bar{\nabla}\bar{r})\Big|^{-1} \delta\Big(U_\phi(\bar{u}, \bar{\phi}, \bar{r}) - u\Big)\bar{X} \ , \tag{6.14}$$

$$D_{\mathbf{1}}^{\rm III}(u,\phi) \;=\; \frac{1}{4\pi}\int d^2\mathcal{S}(\bar\phi)\int\limits_{r_0}^{\infty} d\bar r\bar r^2\int\limits_{u_1}^{u_2} d\bar u\Big|(\bar\nabla\bar u,\bar\nabla\bar r)\Big|^{-1}\delta\Big(U_\phi(\bar u,\bar\phi,\bar r)-u\Big)\bar X \;\;, \quad (6.15)$$

$$D_{\mathbf{1}}^{\rm IV}(u,\phi) \;=\; \frac{1}{4\pi}\int d^2\mathcal{S}(\bar\phi)\int\limits_{r_0}^{\infty} d\bar r\bar r^2\int\limits_{-\infty}^{u_1} d\bar u\Big|(\bar\nabla\bar u,\bar\nabla\bar r)\Big|^{-1}\delta\Big(U_\phi(\bar u,\bar\phi,\bar r)-u\Big)\bar X \;\;. \quad (6.16)$$

The argument u of the function $D_{\mathbf{1}}(u,\phi)$ labels the future light cone of o. As $u\to\infty$, the point o moves along the central geodesic to the future. In addition, the parameters r_0,u_1,u_2 should be made functions of u such that , as $u\to\infty$,

$$\begin{aligned} r_0(u)\to\infty \quad &,\quad u_1(u)\to-\infty \;\;,\;\; u_2(u)\to+\infty \;\;, \\ \frac{r_0(u)}{u}\to 0 \quad &,\quad \frac{u_1(u)}{u}\to 0 \;\;,\;\; \frac{u_2(u)}{u}\to 0 \;\;. \end{aligned} \qquad (6.17)$$

It is then seen from Fig.6 that, as $u\to\infty$, subdomain I shifts to $\mathcal{T}^+$, subdomain II shifts to i^+, subdomain III shifts to $\mathcal{I}^+$, and subdomain IV shifts to i^0. Let us show this by a direct calculation.

Consider first the contribution of the tube, Eq. (6.13). At a however late u the hyperplane $U_\phi(\bar u,\bar\phi,\bar r)=u$ will cross the nonstationary region between the cones $\bar u = u^-$ and $\bar u = u^+$, and, therefore, will not belong entirely to the future Killing domain. Nevertheless, at a sufficiently late u the intersection of the hyperplane with the tube *will* be at $\bar u > u^+$. The null generators of the hyperplane emanating from this intersection to the future will then also be at $\bar u > u^+$. Therefore, in the integral (6.13) with a sufficiently late u one may use Eq. (5.22) owing to which the integrand becomes restricted to

$$\bar u = u - L_\phi(\bar\phi,\bar r) \;\;,\;\; u\to\infty \;\;. \qquad (6.18)$$

Since, in addition, the range of the integration variables $\bar\phi,\bar r$ in (6.13) is compact, $\bar X$ turns out to be at $\mathcal{T}^+$. We obtain

$$D_{\mathbf{1}}^{\rm I}(u,\phi)\Big|_{u\to\infty} = \frac{1}{4\pi}\int d^2\mathcal{S}(\bar\phi)\int\limits_0^{r_0} d\bar r\bar r^2\Big|(\bar\nabla\bar u,\bar\nabla\bar r)\Big|^{-1}\bar X_{\mathcal{T}^+}[\bar r,\bar\phi,\bar u=u\to\infty] \;\;\;, \qquad (6.19)$$

and the geometrical factor in the measure is bounded by virtue of the original assumption that the metric has no horizons.

In the contributions of the domains II, III, IV one can use the flat-spacetime expressions for the metric and the function $U_\phi(\bar u,\bar\phi,\bar r)$, Eqs. (1.15) and (4.15). In the integral (6.14)

go over to the integration variable γ according to the formula

$$\bar{r} = \frac{\gamma}{1-\gamma}\bar{u} \tag{6.20}$$

i.e. parametrize this integral with the family of geodesics (6.5). It follows that, when $u \to \infty$, γ ranges in the interval $0 < \gamma < 1$, and, with γ fixed, $\bar{X}$ turns out to be at i^+. In the integral (6.15), $\bar{u}$ ranges in a bounded interval, and, with $\bar{u}$ fixed, $\bar{X}$ turns out to be at $\mathcal{I}^+$. In the integral (6.16) go over to the integration variable β according to the formula

$$\bar{r} = -\frac{1}{1-\beta}\bar{u} \tag{6.21}$$

i.e. parametrize this integral with the family of geodesics (6.9). Then β ranges in a bounded interval, and $\bar{X}$ turns out to be at i^0. In this way we obtain

$$D_{\mathbf{1}}^{\mathrm{II}}(u,\phi)\Big|_{u\to\infty} = \tag{6.22}$$

$$= \frac{u^3}{4\pi}\int d^2\mathcal{S}(\bar{\phi}) \int\limits_{r_0/u}^{1-(1-\cos\omega)|u_2|/u} d\gamma \, \frac{\gamma^2}{(1-\gamma\cos\omega)^4}\bar{X}_{i^+}[\gamma,\bar{\phi},\bar{s} = \frac{\sqrt{1-\gamma^2}}{1-\gamma\cos\omega}u \to \infty] \ ,$$

$$D_{\mathbf{1}}^{\mathrm{III}}(u,\phi)\Big|_{u\to\infty} = \tag{6.23}$$

$$= \frac{1}{4\pi}\int \frac{d^2\mathcal{S}(\bar{\phi})}{1-\cos\omega}\int\limits_{u_1}^{u_2} d\bar{u}\Big(\bar{r}^2\bar{X}\Big)_{\mathcal{I}^+}[\bar{u},\bar{\phi},\bar{r} = \frac{u}{1-\cos\omega} \to \infty] \ ,$$

$$D_{\mathbf{1}}^{\mathrm{IV}}(u,\phi)\Big|_{u\to\infty} = \tag{6.24}$$

$$= \frac{u}{4\pi}\int d^2\mathcal{S}(\bar{\phi}) \int\limits_{\cos\omega}^{1-(1-\cos\omega)|u_1|/u} d\beta\frac{1}{(\beta-\cos\omega)^2}\Big(\bar{r}^2\bar{X}\Big)_{i^0}[\beta,\bar{\phi},\bar{r} = \frac{u}{\beta-\cos\omega} \to \infty] \ .$$

Here $\omega = \omega(\phi,\bar{\phi})$ is the function of the angles defined in (4.16).

By (6.17), the integration limits in (6.22) reach as $u \to \infty$ the end points of the interval $0 < \gamma < 1$. As explained in Sec.1, the end point $\gamma = 1$ supports the contribution of the future of $\mathcal{I}^+$ and so does the end point $u_2(u) \to \infty$ in (6.23). The behaviours of X at these end points are related by Eq. (1.29). Similarly, the end point $\gamma = 0$ supports the contribution of the boundary of the expanding tube $\mathcal{T}^+$ and so does the end point $r_0(u) \to \infty$ in (6.19). Finally, the upper limit in (6.24) reaches as $u \to \infty$ the end point $\beta = 1$ which supports the contribution of the past of $\mathcal{I}^+$. Another such contribution comes from the end point $u_1(u) \to -\infty$ in (6.23). The end-point contributions are of

measure zero only if the respective integrals converge at these end points; otherwise, they may be essential. (The end point $\beta = -1$ which corresponds to the future of $\mathcal{I}^-$ makes no contribution.)

For simplicity, we may confine ourselves to scalar sources X which are, moreover, stationary in the past:

$$\frac{\partial}{\partial u} X(u, \phi, r) = 0 \ \ , \ \ u < u^- \tag{6.25}$$

since the source I in (6.1) possesses these properties. The behaviour of X at $\mathcal{I}^+$ is then restricted by the convergence condition (4.25). A restriction on the behaviour of X at i^0 can be obtained by considering the retarded Green function with a static source (the Coulomb potential). The convergence condition is in this case

$$X\Big|_{i^0} = O\Big(\frac{1}{r^4}\Big) \ \ . \tag{6.26}$$

Therefore, we may put

$$X_{i^0}[\beta, \phi, r \to \infty] = \frac{A(\phi)}{r^n} \ \ , \ \ \ n \geq 4 \tag{6.27}$$

$$X_{\mathcal{I}^+}[u, \phi, r \to \infty] = \begin{cases} A(\phi)/r^n & , \quad u < u^- \\ B(u, \phi)/r^3 & , \quad u > u^- \end{cases} \tag{6.28}$$

with some coefficients A and B. For X at $\mathcal{T}^+$ and i^+ it suffices to consider the power behaviours

$$X_{\mathcal{T}^+}[r, \phi, u \to \infty] \ = \ C(r, \phi) u^k \ \ , \tag{6.29}$$

$$X_{i^+}[\gamma, \phi, s \to \infty] \ = \ W(\gamma, \phi) s^p \tag{6.30}$$

with arbitrary k and p. However, to agree with (6.28) and (6.29), the coefficient of the latter behaviour should have appropriate singularities at $\gamma = 1$ and $\gamma = 0$:

$$W(\gamma, \phi) = (1 - \gamma)^{3+p/2} \gamma^{p-k} w(\gamma, \phi) \tag{6.31}$$

where $w(\gamma, \phi)$ is a regular function. By the correspondence (1.29) we have also

$$B(u, \phi)\Big|_{u \to \infty} = w(1, \phi) \ 2^{p/2} \ u^{p+3} \ \ , \tag{6.32}$$

and similarly

$$C(r, \phi)\Big|_{r \to \infty} = w(0, \phi) r^{p-k} \ \ . \tag{6.33}$$

Upon the insertion of the behaviours above in (6.19)-(6.24) and the use of (6.17) the following results are obtained. The contribution of i^0 is

$$D_{\mathbf{1}}^{\mathrm{IV}}(u,\phi)\Big|_{u\to\infty} = O(u^{-1}) \ . \tag{6.34}$$

The contribution of $\mathcal{I}^+$ proper, i.e. of a finite range of retarded time along $\mathcal{I}^+$, is also

$$D_{\mathbf{1}}^{\mathrm{III}}(u,\phi)\Big|_{u\to\infty} = O(u^{-1}) \ . \tag{6.35}$$

The contribution of i^+ proper, i.e. of the open interval $0 < \gamma < 1$, is

$$D_{\mathbf{1}}^{\mathrm{II}}(u,\phi)\Big|_{u\to\infty} = O(u^{p+3}) \ . \tag{6.36}$$

Finally, the contribution of the tube proper, i.e. of a tube with compact spatial sections, is

$$D_{\mathbf{1}}^{\mathrm{I}}(u,\phi)\Big|_{u\to\infty} = O(u^{k}) \ . \tag{6.37}$$

As far as the end-point contributions are concerned, they become essential only in the case where the behaviour (6.36) is comparable with (6.35) or (6.37). If the power in (6.36) is different from the one in (6.35) and from the one in (6.37), then the total result is a sum of these competing powers:

$$D_{\mathbf{1}}(u,\phi)\Big|_{u\to\infty} = O(u^{-1}) + O(u^{p+3}) + O(u^{k}) \ , \ \ p+3 \neq -1 \ , \ \ p+3 \neq k \ , \tag{6.38}$$

and one is to pick out the dominant one. If, however, $p+3=-1$ or $p+3=k$, then the joint contribution of the coincident powers gets amplified by a factor of $\log u$:

$$D_{\mathbf{1}}(u,\phi)\Big|_{u\to\infty} = \begin{cases} O(u^{-1}\log u) + O(u^{k}) \ , \ p+3=-1 \\ O(u^{k}\log u) + O(u^{-1}) \ , \ p+3=k \end{cases} . \tag{6.39}$$

A comparison of the behaviours obtained with the one in (6.2) shows that relevant contributions may come only from X at i^+ or $\mathcal{T}^+$. However, at i^+, there are three extra powers added to the exponent of the behaviour of X at late time, Eq. (6.36). If X doesn't grow at either of the limits (as is normally the case) or has one and the same power of growth at i^+ and $\mathcal{T}^+$, then only X at i^+ is capable of making a contribution to the real vacuum energy. Therefore, the most important case is generally the one where the contribution of X at i^+ is dominant:

$$p > -4 \ , \ \ p > k-3 \ . \tag{6.40}$$

In this case the result is

$$D_{\mathbf{1}}(u,\phi|X)\Big|_{u\to\infty} = \frac{u^3}{4\pi}\int d^2\mathcal{S}(\bar\phi)\int\limits_0^1 d\gamma\frac{\gamma^2}{(1-\gamma\cos\omega)^4}\bar{X}_{i^+}[\gamma,\bar\phi,\bar{s}=\frac{u\sqrt{1-\gamma^2}}{1-\gamma\cos\omega}\to\infty] \tag{6.41}$$

with $\cos\omega$ in (4.16). With the specification in (6.31) this integral converges.

Formula for the energy of vacuum radiation.

It will be shown below that for $X = I$ the conditions (6.40) are fulfilled with $p = k$, and the exponent p in (6.30) has the needed value $p = -1$:

$$I_{i^+}[\gamma,\phi,s\to\infty] = \frac{1}{s}\,W(\gamma,\phi)\ . \tag{6.42}$$

Then Eq. (6.41) yields

$$D_{\mathbf{1}}(u,\phi|I)\Big|_{u\to\infty} = \frac{u^2}{4\pi}\int d^2\mathcal{S}(\bar\phi)\int\limits_0^1 d\gamma\frac{\gamma^2}{\sqrt{1-\gamma^2}}\Big(1-\gamma\cos\omega(\phi,\bar\phi)\Big)^{-3}W(\gamma,\bar\phi)\ . \tag{6.43}$$

According to (6.1), the coefficient of u^2 in this expression is the angle distribution of the vacuum radiation.

The integration over the directions of radiation, i.e. over the angles ϕ at $\mathcal{I}^+$ [16], can be done explicitly:

$$\int d^2\mathcal{S}(\phi)\Big(1-\gamma\cos\omega(\phi,\bar\phi)\Big)^{-3} = \frac{4\pi}{(1-\gamma^2)^2} \tag{6.44}$$

whence for the total radiation energy (6.1) one obtains the result

$$M(-\infty)-M(\infty) = -\frac{1}{4\pi^2}\int\limits_0^1 d\gamma\gamma^2\int d^2\mathcal{S}(\phi)\frac{W(\gamma,\phi)}{(1-\gamma^2)^{5/2}} \tag{6.45}$$

in terms of the coefficient in (6.42). Eq. (6.31) for the case (6.42) is of the form

$$W(\gamma,\phi) = (1-\gamma)^{5/2}w(\gamma,\phi) \tag{6.46}$$

[16] Not to be confused with the angles ϕ at i^+ on which the function $W(\gamma,\phi)$ in (6.42) depends. A summary table of the integrals over the 2-sphere used in the paper is given in Appendix B.

which ensures the convergence of the integral (6.45). The coefficient $W(\gamma,\phi)$ in (6.42) will be calculated and shown to be negative definite.

It will be recalled that condition (6.46) emerges from the correspondence between the limits i^+ and $\mathcal{I}^+$ and is a consequence of the convergence condition (4.26) at $\mathcal{I}^+$. However, this correspondence concerns only the future of $\mathcal{I}^+$. Therefore, for the validity of (6.46) and hence for the validity of the result (6.45) it is only important that the convergence condition at $\mathcal{I}^+$ hold at late time. There should be a value u' of retarded time, no mattar how late, such that

$$\Bigl(I\Bigr|_{\mathcal{I}^+[u,\phi,r\to\infty]}\Bigr)_{u>u'} = O\Bigl(\frac{1}{r^3}\Bigr) \quad . \tag{6.47}$$

The vertices and trees in the future of $\mathcal{I}^+$.

The analysis above of the behaviour of $D_{\mathbf{1}}(u,\phi|X)$ as $u\to\infty$ will now be applied to the case $X=J$ where J is any of the physical sources (1.8). Since the support of J is confined to a tube, of the four contributions (6.34)-(6.37) there remains only the contribution of $\mathcal{T}^+$, Eq. (6.37). Since, at $\mathcal{T}^+$, J is stationary, we obtain immediately

$$D_{\mathbf{1}}(u,\phi|J)\Bigr|_{u\to\infty} = O(u^0) \quad . \tag{6.48}$$

This result can be made more precise. Let u' be the value of retarded time such that for $u>u'$ the intersection of the hyperplane $U_\phi(\bar{u},\bar{\phi},\bar{r})=u$ with the support tube of J is entirely in the future Killing domain $\bar{u}>u^+$. Then for $u>u'$ one may repeat the calculation in (5.37)-(5.38) to obtain by (5.2)

$$\frac{\partial}{\partial u}D_{\mathbf{1}}(u,\phi|J)\Bigr|_{u>u'} = 0 \quad . \tag{6.49}$$

Eq. (6.49) determines the behaviours of the vertex functions in the future of $\mathcal{I}^+$ since it provides a cut off for the time integrals. Thus from (4.29) we obtain

$$\begin{aligned}&\Bigl(F(m,n)J_1J_2\Bigr|_{\mathcal{I}^+[u,\phi,r\to\infty]}\Bigr)_{u\to\infty} = \\ &= 2(-1)^{m+n+1}\frac{u^{m+n}}{r^3}\int\limits_{-\infty}^{\infty}d\tau\Bigl[\Bigl(\frac{d}{d\tau}\Bigr)^{m+1}D_{\mathbf{1}}(\tau,\phi|J_1)\Bigr]\Bigl[\Bigl(\frac{d}{d\tau}\Bigr)^{n+1}D_{\mathbf{1}}(\tau,\phi|J_2)\Bigr] \ .\end{aligned} \tag{6.50}$$

Specifically, the behaviour of the vertex function $\hat{V}_{\rm scalar}$ in (2.43) is

$$\Big(\hat{V}_{\rm scalar}\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]}\Big)_{u\to\infty} = -\frac{u^2}{r^3}\int\limits_{-\infty}^{\infty} d\tau \Big[\frac{d^2}{d\tau^2}\hat{D}^Q_{\mathbf{1}}(\tau,\phi)\Big]\Big[\frac{d^2}{d\tau^2}\hat{D}^Q_{\mathbf{1}}(\tau,\phi)\Big] \tag{6.51}$$

where the abbreviation (4.36) is used. In the same way, for $\hat{V}_{\rm cross}$ we obtain from (4.64)

$$\Big(\nabla_\alpha\nabla_\beta\hat{V}^{\alpha\beta}_{\rm cross}\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]}\Big)_{u\to\infty} = -\frac{1}{2}\frac{u^2}{r^3}\int\limits_{-\infty}^{\infty} d\tau \Big[\frac{d^2}{d\tau^2}D^R_{\mathbf{1}}(\tau,\phi)\Big]\Big[\frac{d^2}{d\tau^2}\hat{D}^Q_{\mathbf{1}}(\tau,\phi)\Big]\ . \tag{6.52}$$

More important are, however, the behaviours of the vector and tensor vertices since they are $O(1/r^2)$ at $\mathcal{I}^+$. For $\hat{V}_{\rm vect}$ we have from (4.65)

$$\Big(\nabla_\alpha\nabla_\beta\hat{V}^{\alpha\beta}_{\rm vect}\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]}\Big)_{u\to\infty} = \frac{1}{6}\frac{u}{r^2}\int\limits_{-\infty}^{\infty} d\tau\, g_{\alpha\beta}\Big[\frac{d}{d\tau}\hat{D}^\alpha_{\mathbf{1}}(\tau,\phi)\Big]\Big[\frac{d}{d\tau}\hat{D}^\beta_{\mathbf{1}}(\tau,\phi)\Big]+O\Big(\frac{1}{r^3}\Big) \tag{6.53}$$

but the result for $\hat{V}_{\rm grav}$ is completely different:

$$\Big(\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu\hat{V}^{\alpha\beta\mu\nu}_{\rm grav}\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]}\Big)_{u>u'} = O\Big(\frac{1}{r^3}\Big) \tag{6.54}$$

owing to (4.81) and (6.49). The latter result applies also to the only tree that is $O(1/r^2)$ at $\mathcal{I}^+$, Eq. (4.101).

Inspecting Eqs. (2.37)-(2.39) one infers that the behaviours obtained here and in Sec.4 can be summarized as the following result for the scalar I:

$$\Big(I\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]}\Big)_{u>u'} = \frac{1}{6r^2}\int\limits_{-\infty}^{u} d\tau(u-\tau){\rm tr}\Big[g_{\alpha\beta}\Big(\frac{d}{d\tau}\hat{D}^\alpha_{\mathbf{1}}\Big)\Big(\frac{d}{d\tau}\hat{D}^\beta_{\mathbf{1}}\Big)\Big] + O\Big(\frac{1}{r^3}\Big) \tag{6.55}$$

where the only $1/r^2$ term at late time comes from the vector source. Since, in (6.55), $\tau\le u$ and

$${\rm tr}\Big[g_{\alpha\beta}\Big(\frac{d}{d\tau}\hat{D}^\alpha_{\mathbf{1}}\Big)\Big(\frac{d}{d\tau}\hat{D}^\beta_{\mathbf{1}}\Big)\Big]\le 0 \tag{6.56}$$

(see Sec.8), the integrand is negative definite. The only possibility for vanishing of the $1/r^2$ term in (6.55) is, therefore,

$${\rm tr}\Big[g_{\alpha\beta}\Big(\frac{d}{d\tau}\hat{D}^\alpha_{\mathbf{1}}\Big)\Big(\frac{d}{d\tau}\hat{D}^\beta_{\mathbf{1}}\Big)\Big]\equiv 0\ . \tag{6.57}$$

By imposing this limitation on the vector source we force condition (6.47) to hold.

The trees at i^+ and $\mathcal{T}^+$.

The inference from the analysis above is that the real vacuum energy is determined by the behaviour of the scalar I at i^+ and possibly at $\mathcal{T}^+$. We begin the study of these behaviours with the trees.

The only nonlocal object that figures in the trees is $(1/\Box)J(x)$ with $1/\Box$ in (3.13) and $J(x)$ in (1.8). The past light cone of x with x moving to i^+ or $\mathcal{T}^+$ is similar to the null hyperplane shifted to the future in a sense that there exists a value u' of retarded time such that for $u(x) > u'$ the intersection of the past light cone of x with the support tube of J is entirely in the future Killing domain. Then also the null geodesics connecting the points of this intersection with the vertex x of the cone are entirely in the Killing domain. Therefore, at $u(x) > u'$ one may use Eqs. (5.32) and (5.2) to obtain

$$\mathcal{L}_\xi \frac{1}{\Box} J(x) = 0 \ \ , \ \ u(x) > u' \ \ . \tag{6.58}$$

For the behaviour at $\mathcal{T}^+$ this gives straight away

$$\frac{1}{\Box} J\Big|_{\mathcal{T}^+[r,\phi,u\to\infty]} = O(u^0) \ \ . \tag{6.59}$$

As far as the behaviour at i^+ is concerned, it is the same as the behaviour at $\mathcal{I}^+$ for $u(x) > u'$ since, in the static case, there is no dependence on the direction at infinity. In this way we obtain

$$\frac{1}{\Box} J\Big|_{i^+[\gamma,\phi,s\to\infty]} = O\Big(\frac{1}{s}\Big) \ \ . \tag{6.60}$$

It follows that at $\mathcal{T}^+$ all the trees (4.82),(4.84) and (4.96) are $O(u^0)$. At i^+, the trees (4.82) vanish identically, and the trees (4.84) and (4.96) are $O(1/s^2)$. In terms of Eqs. (6.29) and (6.30) the maximum exponents of the trees are, therefore, $k = 0$ and $p = -2$ whereas for making a nonvanishing contribution to the real vacuum energy these exponents should be $k \geq 2$, $p \geq -1$. Thus the contribution of the trees vanishes.

Having excluded the trees we may go over to the chief thing: the behaviours of the vertex functions at i^+.

7 The late-time behaviours (continued)

Spacelike hyperplanes.

The behaviour of the world function as one of its points tends to i^+ and the other one stays in a compact domain can be obtained as follows. Since $\sqrt{-2\sigma(x,\bar{x})}$ is the geodetic distance between x and $\bar{x}$, in the principal approximation as $x \to i^+$ there should be

$$\sqrt{-2\sigma(x,\bar{x})}\Big|_{x\to i^+} = s(x)\Big(1+\mathcal{O}\Big) \ \ , \ \ \mathcal{O}\Big|_{i^+} = 0 \tag{7.1}$$

where $s(x) \to \infty$ is the proper time of the point x moving to i^+. In a curved spacetime there is one more growing term, proportional to $\log s(x)$. Therefore, generally we have an expansion of the form

$$\sqrt{-2\sigma(x,\bar{x})}\Big|_{x\in i^+[\gamma,\phi,s\to\infty]} = c(s) - \frac{T_{\gamma\phi}(\bar{x})}{\sqrt{1-\gamma^2}} + O\Big(\frac{1}{s}\Big) \ \ , \tag{7.2}$$

$$c(s) = s + O(\log s) \tag{7.3}$$

where $c(s)$ is restricted by an additional condition that it doesn't depend on $\bar{x}$, and a notation is introduced for the $O(s^0)$ term of the expansion. In this term, $T_{\gamma\phi}(\bar{x})$ is some function of the point $\bar{x}$ depending also on the coordinates γ, ϕ of the point x, and the normalization factor $\sqrt{1-\gamma^2}$ is introduced for further convenience. The $T_{\gamma\phi}(\bar{x})$ is defined up to an addition of an arbitrary finite function of x i.e. a function of γ, ϕ.

The insertion of the asymptotic behaviour (7.2) in the equation (3.2) for σ with respect to the point $\bar{x}$ yields

$$\Big(\bar{\nabla}T(\bar{x})\Big)^2 = -(1-\gamma^2) \quad , \quad T(\bar{x}) \equiv T_{\gamma\phi}(\bar{x}) \ \ , \tag{7.4}$$

and, since $\bar{\nabla}\sigma(x,\bar{x})$ is past directed, so is $\bar{\nabla}T(\bar{x})$. The function $T_{\gamma\phi}(x)$ with fixed γ and ϕ defines the family of spacelike hypersurfaces

$$T_{\gamma\phi}(x) = \tau = \text{const.} \tag{7.5}$$

with time τ growing towards the future. The vector field orthogonal to these hypersurfaces

$$N^\alpha(x) = \nabla^\alpha \frac{T_{\gamma\phi}(x)}{\sqrt{1-\gamma^2}} \quad , \quad N^2(x) = -1 \tag{7.6}$$

is a gradient and at the same time has a unit norm. A combination of these two properties signifies that the integral curves of this vector field

$$\frac{dx^\alpha}{ds} = -N^\alpha(x) \tag{7.7}$$

are geodesics. The specification of these geodesics can be read off from Eq.(7.2):

$$-N^\alpha(\bar{x}) = \bar{\nabla}^\alpha \sqrt{-2\sigma(x,\bar{x})}\Big|_{x\in i^+[\gamma,\phi,s\to\infty]} \quad . \tag{7.8}$$

At every point $\bar{x}$ the vector field $N^\alpha(\bar{x})$ is tangent to a timelike geodesic that, when traced to the future, appears at the asymptotically flat infinity with the energy $E = (1-\gamma^2)^{-1/2}$ at the point ϕ of the celestial sphere. In a compact domain the geodesics having one and the same γ and ϕ differ by "translations" and make a 3-parameter congruence. It follows from (7.2) that the geodesic congruence thus defined is hypersurface-orthogonal, and the orthogonal hypersurfaces are just (7.5). The hypersurfaces (7.5) will be called hyperplanes, and there are different families of spacelike hyperplanes for different values of γ and ϕ.[17]

By using Eq. (6.5), the expansion (7.2) can be rewritten in terms of $r(s) \to \infty$:

$$\sqrt{-2\sigma(x,\bar{x})}\Big|_{x\in i^+[\gamma,\phi,s\to\infty]} = \frac{\sqrt{1-\gamma^2}}{\gamma}\Big(r + O(\log r)\Big) - \frac{T_{\gamma\phi}(\bar{x})}{\sqrt{1-\gamma^2}} + O\Big(\frac{1}{r}\Big) \tag{7.9}$$

whence, going over to the limit $\gamma \to 1$, we obtain

$$\Big(\sigma(x,\bar{x})\Big|_{x\in i^+[\gamma,\phi,s\to\infty]}\Big)_{\gamma\to 1} = r\Big(T_{\gamma\phi}(\bar{x})\Big|_{\gamma=1} + \mathcal{O}\Big) \ , \ \ \mathcal{O}\Big|_{i^+} = 0 \ . \tag{7.10}$$

By the correspondence (1.29), this sequence of limits should coincide with the future of $\mathcal{I}^+$. On the other hand, at $\mathcal{I}^+$ we have the expansion (4.5) in which the function $U_\phi(\bar{x})$ is independent of the time along $\mathcal{I}^+$. It follows that

$$T_{\gamma\phi}(\bar{x})\Big|_{\gamma=1} = U_\phi(\bar{x}) + \text{const.} \tag{7.11}$$

where the const. is independent of $\bar{x}$. Thus we infer that the function $T_{\gamma\phi}(x)$ admits the limit $\gamma \to 1$ and, at this limit, the spacelike hyperplanes (7.5) turn into the null

[17]The Bondi and ADM masses refer to the Lorentz frame at infinity in which the asymptotically flat spacetime rests as a whole, i.e. the center-of-mass frame. The parameter γ refers to the same frame. Therefore, the dependence on γ cannot be boosted away and is real.

hyperplanes (4.7). The geodesics orthogonal to the spacelike hyperplanes also become null and turn into the generators of the null hyperplanes.

The insertion of expansion (7.2) in (3.4) yields

$$\bar{\nabla}_{\bar{\mu}} \frac{T_{\gamma\phi}(\bar{x})}{\sqrt{1-\gamma^2}} = g_{\bar{\mu}}{}^{\mu}(\bar{x}, x) \nabla_\mu s(x) \Big|_{x \to i^+[\gamma,\phi]} \tag{7.12}$$

which is the law of parallel transport of the tangent vector along a timelike geodesic orthogonal to the hyperplanes. By using the asymptotic form (6.5) of the geodesic, one may expand ∇s over the vector basis at i^+ :

$$\nabla_\mu s(x) = \frac{1}{\sqrt{1-\gamma^2}} \Big(\nabla_\mu t(x) - \gamma \nabla_\mu r(x) \Big) \ , \quad x \to i^+[\gamma, \phi] \ . \tag{7.13}$$

Then the law of parallel transport takes the form

$$\bar{\nabla}_{\bar{\mu}} T_{\gamma\phi}(\bar{x}) = g_{\bar{\mu}}{}^{\mu}(\bar{x}, x) \Big(\nabla_\mu t(x) - \gamma \nabla_\mu r(x) \Big) \Big|_{x \to i^+[\gamma,\phi]} \tag{7.14}$$

and at $\gamma = 1$ goes over into (4.11).

In presence of a timelike Killing vector the spacelike hyperplanes possess the properties similar to (5.20) and (5.21):

$$\bar{\xi}^\mu \bar{\nabla}_\mu T_{\gamma\phi}(\bar{x}) = 1 \ , \tag{7.15}$$

$$\mathcal{L}_{\bar{\xi}} \bar{\nabla}_{\alpha_1} \ldots \bar{\nabla}_{\alpha_n} T_{\gamma\phi}(\bar{x}) = 0 \ , \quad n \geq 1 \ . \tag{7.16}$$

Finally, in the case where $\bar{x}$ is at the future asymptotically flat infinity $\mathcal{I}^+$ or i^+, one can use the flat-spacetime formula for $T_{\gamma\phi}(\bar{x})$:

$$\begin{aligned} T_{\gamma\phi}(\bar{x}) \Big|_{\bar{x} \to \mathcal{I}^+, i^+} &= \bar{u} + \bar{r} \Big(1 - \gamma \cos \omega(\phi, \bar{\phi}) \Big) \\ &= \bar{t} - \gamma n_i(\phi) \bar{\mathbf{x}}^i \end{aligned} \tag{7.17}$$

with the same notation as in (4.15).

The radiation moments and conserved charges.

The function $T_{\gamma\phi}(x)$ may formally be regarded as a two-point function with one point at i^+:

$$T_{\gamma\phi}(\bar{x}) \equiv T(x, \bar{x}) \Big|_{x \to i^+[\gamma,\phi]} \ . \tag{7.18}$$

With the aid of this notation and with the sources in (1.8), define the following *radiation moments*:

$$\hat{D}(x)\Big|_{i^+} = \frac{1}{4\pi}\int d\bar{x}\bar{g}^{1/2}\delta\Big(T(x,\bar{x})-\tau\Big)\hat{P}(\bar{x})\Big|_{x\to i^+} \ , \tag{7.19}$$

$$\hat{D}^{\alpha}(x)\Big|_{i^+} = \frac{1}{4\pi}\int d\bar{x}\bar{g}^{1/2}\delta\Big(T(x,\bar{x})-\tau\Big)g^{\alpha}{}_{\bar{\alpha}}(x,\bar{x})\hat{J}^{\bar{\alpha}}(\bar{x})\Big|_{x\to i^+} \ , \tag{7.20}$$

$$D^{\alpha\beta}(x)\Big|_{i^+} = \frac{1}{4\pi}\int d\bar{x}\bar{g}^{1/2}\delta\Big(T(x,\bar{x})-\tau\Big)g^{\alpha}{}_{\bar{\alpha}}(x,\bar{x})g^{\beta}{}_{\bar{\beta}}(x,\bar{x})J^{\bar{\alpha}\bar{\beta}}(\bar{x})\Big|_{x\to i^+} \ . \tag{7.21}$$

These objects are respectively a matrix, a matrix vector, and a tensor defined, however, only for x at i^+. The specification i^+ will, therefore, be omitted and we shall write simply $\hat{D}, \hat{D}^{\alpha}, D^{\alpha\beta}$. The similarly defined moments for the Ricci scalar R and the matrix $\hat{Q}$ will be denoted D^R and $\hat{D}^Q$. In addition to the parameters γ and ϕ of the point at i^+, the radiation moments depend on the parameter τ of the hyperplane. As follows from (7.11), the previously introduced moments $D_{\mathbf{1}}$ are the special cases of the moments D corresponding to $\gamma = 1$:

$$\hat{D}_{\mathbf{1}} = \hat{D}\Big|_{\gamma=1} \ , \quad \hat{D}_{\mathbf{1}}^{\alpha} = \hat{D}^{\alpha}\Big|_{\gamma=1} \ , \quad D_{\mathbf{1}}^{\alpha\beta} = D^{\alpha\beta}\Big|_{\gamma=1} \ , \quad \text{etc.} \tag{7.22}$$

Owing to the law of parallel transport (7.14) and the conservation laws (1.9), the vector and tensor moments satisfy the relations

$$(\nabla_{\alpha}t-\gamma\nabla_{\alpha}r)\frac{d}{d\tau}\hat{D}^{\alpha} = 0 \ , \tag{7.23}$$

$$(\nabla_{\alpha}t-\gamma\nabla_{\alpha}r)\frac{d}{d\tau}D^{\alpha\beta} = 0 \tag{7.24}$$

which generalize (4.42) and (4.48), and are proved similarly. The conserved quantities in these relations

$$\hat{e} = (\nabla_{\mu}t-\gamma\nabla_{\mu}r)\hat{D}^{\mu} = \frac{1}{4\pi}\int d\bar{x}\bar{g}^{1/2}\delta\Big(T(x,\bar{x})-\tau\Big)\bar{\nabla}_{\bar{\mu}}T\hat{J}^{\bar{\mu}}(\bar{x})\Big|_{x\to i^+} \tag{7.25}$$

and

$$p^{\nu} = \frac{1}{2}(\nabla_{\mu}t-\gamma\nabla_{\mu}r)D^{\mu\nu} = \frac{1}{8\pi}\int d\bar{x}\bar{g}^{1/2}\delta\Big(T(x,\bar{x})-\tau\Big)g^{\nu}{}_{\bar{\nu}}(x,\bar{x})\bar{\nabla}_{\bar{\mu}}TJ^{\bar{\mu}\bar{\nu}}(\bar{x})\Big|_{x\to i^+} \tag{7.26}$$

are the full charges (4.44) and (4.50) since the latter are independent of the choice of the hypersurface $\Sigma = 0$ crossing the support tube of J.

By using (3.6) and the conservation laws, the vector and tensor moments can be brought to the forms

$$\hat{D}^\alpha = \frac{d}{d\tau}\hat{\mathcal{D}}^\alpha \quad , \quad D^{\alpha\beta} = \frac{d^2}{d\tau^2}\mathcal{D}^{\alpha\beta} \tag{7.27}$$

with

$$\hat{\mathcal{D}}^\alpha(x)\Big|_{i^+} = -\frac{1}{4\pi}\int d\bar{x}\bar{g}^{1/2}\delta\Big(T(x,\bar{x})-\tau\Big)\sigma^\alpha(x,\bar{x})\bar{\nabla}_{\bar{\mu}}T\hat{J}^{\bar{\mu}}(\bar{x})\Big|_{x\to i^+} \quad , \tag{7.28}$$

$$\mathcal{D}^{\alpha\beta}(x)\Big|_{i^+} = \frac{1}{8\pi}\int d\bar{x}\bar{g}^{1/2}\delta\Big(T(x,\bar{x})-\tau\Big)\sigma^\alpha(x,\bar{x})\sigma^\beta(x,\bar{x})\bar{\nabla}_{\bar{\mu}}T\bar{\nabla}_{\bar{\nu}}TJ^{\bar{\mu}\bar{\nu}}(\bar{x})\Big|_{x\to i^+} \quad . \tag{7.29}$$

Although we called moments the D 's, this name is more appropriate for the $\mathcal{D}$'s. Their relation to the textbook multipole moments is considered in Sec.8. The $\mathcal{D}$'s always appear differentiated with respect to τ. Undifferentiated, they are infinite at the limit i^+ as distinct from the D 's. The consideration below is carried out in terms of the D 's in (7.19)-(7.21).

The vertex operators at i^+ .

An essential point concerning the vertex functions is that the hyperboloid $\sigma(x,\bar{x}) = q$ with q ranging from 0 to $-\infty$ sweeps the whole causal past of x. As $x \to i^+$, the hyperboloid of x with any fixed q will go out of the support of the nonstationary sources but there will always be a range of q for which the hyperboloid crosses this support. For q to stay in this range, it should be shifted to $-\infty$ simultaneously with $x \to i^+$. This suggests the following replacement of the integration variable in (3.23), (3.32) and (3.33):

$$\sqrt{-2q} = c(s) - \frac{\tau}{\sqrt{1-\gamma^2}} \tag{7.30}$$

where τ is the new integration variable, $c(s)$ is the function in (7.3), and γ, s are the parameters of $x \in i^+[\gamma,\phi,s\to\infty]$. With q replaced as in (7.30) and τ fixed, we obtain from (7.2)

$$\delta\Big(\sigma(x,\bar{x})-q\Big)\Big|_{x\in i^+[\gamma,\phi,s\to\infty]} = \frac{\sqrt{1-\gamma^2}}{s}\delta\Big(T_{\gamma\phi}(\bar{x})-\tau\Big) \quad . \tag{7.31}$$

Hence

$$\mathcal{H}_q X(x)\Big|_{x\in i^+[\gamma,\phi,s\to\infty]} = \frac{\sqrt{1-\gamma^2}}{s}D_\gamma(\tau,\phi|X) \tag{7.32}$$

where

$$D_\gamma(\tau,\phi|X)=\frac{1}{4\pi}\int d\bar{x}\bar{g}^{1/2}\delta\Big(T_{\gamma\phi}(\bar{x})-\tau\Big)\bar{X} \tag{7.33}$$

is the full (γ- dependent) radiation moment of the test source X. The moment in (4.19) is its limiting case

$$D_{\mathbf{1}}(\tau,\phi|X)=D_\gamma(\tau,\phi|X)\Big|_{\gamma=1} \quad . \tag{7.34}$$

Another essential point concerns the spacelike hyperplane (7.5) at late time τ. This case is analogous to the case of the null hyperplane. As the hyperplane (7.5) shifts to the future, its intersection with the support tube of J turns out to be in the future Killing domain as well as the timelike geodesics emanating from this intersection towards i^+. Eq. (7.15) can then be used to obtain for the spacelike hyperplane the result analogous to (6.49). Namely, there exists an instant τ' of time τ such that

$$\frac{d}{d\tau}D_\gamma(\tau,\phi|J)\Big|_{\tau>\tau'}=0 \quad . \tag{7.35}$$

One is now ready for obtaining the vertex functions at i^+. Making the replacement (7.30) in the integral (3.23), and using (7.32) and (7.35) one arrives at the following result:

$$F(m,n)J_1J_2\Big|_{i^+[\gamma,\phi,s\to\infty]}= \tag{7.36}$$

$$=s^{m+n-3}\Big(-\frac{1}{2}\Big)^{m+n-1}\Big(\sqrt{1-\gamma^2}\Big)^{m+n+3}\int\limits_{-\infty}^{\infty}d\tau\Big[\Big(\frac{d}{d\tau}\Big)^{m+1}D_\gamma(\tau,\phi|J_1)\Big]\Big[\Big(\frac{d}{d\tau}\Big)^{n+1}D_\gamma(\tau,\phi|J_2)\Big] \quad .$$

This is the sought for growth in time. Comparing this result with Eq. (6.50) one can check the fulfillment of the correspondence (1.29) between the limits i^+ and $\mathcal{I}^+$. A similar calculation for the operators (3.32) and (3.33) yields

$$\frac{1}{\Box_2}F(m,n)J_1J_2\Big|_{i^+[\gamma,\phi,s\to\infty]}= \tag{7.37}$$

$$=-s^{m+n-3}\Big(-\frac{1}{2}\Big)^{m+n-1}\Big(\sqrt{1-\gamma^2}\Big)^{m+n+1}\int\limits_{-\infty}^{\infty}d\tau\Big[\Big(\frac{d}{d\tau}\Big)^{m+1}D_\gamma(\tau,\phi|J_1)\Big]\Big[\Big(\frac{d}{d\tau}\Big)^{n-1}D_\gamma(\tau,\phi|J_2)\Big] \quad ,$$

$$\frac{1}{\Box_1\Box_2}F(m,n)J_1J_2\Big|_{i^+[\gamma,\phi,s\to\infty]}= \tag{7.38}$$

$$=s^{m+n-3}\Big(-\frac{1}{2}\Big)^{m+n-1}\Big(\sqrt{1-\gamma^2}\Big)^{m+n-1}\int\limits_{-\infty}^{\infty}d\tau\Big[\Big(\frac{d}{d\tau}\Big)^{m-1}D_\gamma(\tau,\phi|J_1)\Big]\Big[\Big(\frac{d}{d\tau}\Big)^{n-1}D_\gamma(\tau,\phi|J_2)\Big] \quad .$$

Specializing to the vertex functions in (2.40)-(2.43) one obtains

$$\hat{V}_{\rm scalar}\Big|_{i^+[\gamma,\phi,s\to\infty]} = -\frac{1}{4s}(1-\gamma^2)^{5/2}\int\limits_{-\infty}^{\infty} d\tau\Big(\frac{d^2}{d\tau^2}\hat{D}^Q\Big)\Big(\frac{d^2}{d\tau^2}\hat{D}^Q\Big) \ , \tag{7.39}$$

$$\hat{V}^{\alpha\beta}_{\rm cross}\Big|_{i^+[\gamma,\phi,s\to\infty]} = \frac{s}{48}(1-\gamma^2)^{5/2}\int\limits_{-\infty}^{\infty} d\tau\Big(\frac{d^2}{d\tau^2}D^{\alpha\beta}\Big)\Big(\frac{d^2}{d\tau^2}\hat{D}^Q\Big) \ , \tag{7.40}$$

$$\hat{V}^{\alpha\beta}_{\rm vect}\Big|_{i^+[\gamma,\phi,s\to\infty]} = \frac{s}{48}(1-\gamma^2)^{3/2}\int\limits_{-\infty}^{\infty} d\tau\Big(\frac{d}{d\tau}\hat{D}^\alpha\Big)\Big(\frac{d}{d\tau}\hat{D}^\beta\Big) \ , \tag{7.41}$$

$$\hat{V}^{\alpha\beta\mu\nu}_{\rm grav}\Big|_{i^+[\gamma,\phi,s\to\infty]} = -\hat{1}\frac{s^3}{180\times 32}(1-\gamma^2)^{5/2}\int\limits_{-\infty}^{\infty} d\tau\Big(\frac{d^2}{d\tau^2}D^{\alpha\beta}\Big)\Big(\frac{d^2}{d\tau^2}D^{\mu\nu}\Big) \ . \tag{7.42}$$

In each case, only the highest exponents m and n of the vertex operators $F(m,n)$ work.

As seen from the expressions above, the function $\hat{V}_{\rm scalar}$ has precisely the needed power of growth at i^+ (cf. Eq. (6.42)) but the remaining functions grow apparently too fast. On the other hand, in Eq. (2.39) these latter functions appear differentiated. Unlike in the case of $\mathcal{I}^+$, here the derivatives alone help as we show below. However, the conservation laws simplify the result greately.

The non-scalar vertices at i^+ .

In view of (7.13), the conservation laws (7.23) and (7.24) for the moments can be written in the form

$$\nabla_\alpha s\,\frac{d}{d\tau}\hat{D}^\alpha = 0 \quad , \quad \nabla_\alpha s\,\frac{d}{d\tau}D^{\alpha\beta} = 0 \ . \tag{7.43}$$

Therefore, when projected on ∇s, the senior asymptotic terms of the vertex functions (7.40)-(7.42) vanish. For further use we shall record this fact in the form

$$\nabla_\alpha s\, V^{\alpha\ldots}\Big|_{i^+} = O\Big(\frac{1}{s}V^{\alpha\ldots}\Big) \tag{7.44}$$

where $V^{\alpha\ldots}$ is any of the vertex functions bearing indices [18].

[18] When referring to tensors at infinity we always mean their components in the null-tetrad basis or, equivalently, in the Minkowski frame. The components of radiation moments in this frame are finite.

For tensors $V^{\alpha\beta}$ and $V^{\alpha\beta\mu\nu}$ possessing the property (7.44) we are to calculate the behaviours at i^+ of the quantities

$$\nabla_\alpha\nabla_\beta V^{\alpha\beta} \quad , \quad \nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu V^{\alpha\beta\mu\nu} \tag{7.45}$$

appearing in (2.39). For this purpose it is convenient to go over to new coordinates of the point at i^+:

$$x\Big|_{i^+} = (s,\gamma,\phi) = (s,\gamma^i) \ , \ \ \gamma^i = \gamma n^i(\phi) \tag{7.46}$$

where s,γ^i are the new coordinates, and $n^i(\phi)$ is defined in (1.27). The coordinates s,γ^i can be related also to the Minkowski coordinates at i^+. In the Minkowski frame (1.25) the geodesic (6.5) is of the form

$$t = \frac{s}{\sqrt{1-\gamma^2}} \ , \ \ \mathbf{x}^i = \frac{\gamma^i s}{\sqrt{1-\gamma^2}} \ , \ \ s\to\infty \tag{7.47}$$

which gives the transformation law.

The transformation law (7.47) makes it easy to obtain the metric at i^+ in the new frame. We have

$$\begin{aligned} g^{00} \equiv (\nabla s,\nabla s) = -1 \quad &, \quad g^{0i} \equiv (\nabla s,\nabla\gamma^i) = 0 \ , \\ g^{ik} \equiv (\nabla\gamma^i,\nabla\gamma^k) = \frac{1}{s^2}\tilde{g}^{ik} \quad &, \quad \tilde{g}^{ik}\tilde{g}_{kp} = \delta^i_p \end{aligned} \tag{7.48}$$

where $\tilde{g}_{ik}$ is the 3-dimensional metric depending only on γ^i:

$$\begin{aligned} \tilde{g}_{ik} = \frac{1}{1-\gamma^2}\Big(\delta_{ik} + \frac{\gamma_i\gamma_k}{1-\gamma^2}\Big) \quad &, \quad \gamma_i = \delta_{ik}\gamma^k \ , \\ \tilde{g}^{ik} = (1-\gamma^2)(\delta^{ik} - \gamma^i\gamma^k) \quad &, \quad \sqrt{\tilde{g}} \equiv \sqrt{\det\tilde{g}_{ik}} = \frac{1}{(1-\gamma^2)^2} \ . \end{aligned} \tag{7.49}$$

Covariant derivatives with respect to the 3-metric $\tilde{g}_{ik}$ will be denoted $\tilde{\nabla}_i$, and the following relation will be quoted:

$$\tilde{\nabla}_i\tilde{\nabla}_k\frac{1}{\sqrt{1-\gamma^2}} = \frac{1}{\sqrt{1-\gamma^2}}\,\tilde{g}_{ik} \ . \tag{7.50}$$

It will also be noted that, in terms of γ^i, the integral over the parameters at i^+ that figures in (6.45) is of the form

$$\int_0^1 d\gamma\gamma^2\int d^2\mathcal{S}(\phi)X\Big|_{i^+[\gamma,\phi,s\to\infty]} = \int d^3\gamma\,\theta(1-\gamma)X\Big|_{i^+[\gamma^i,s\to\infty]} \ . \tag{7.51}$$

By a direct calculation in the metric (7.48) one obtains

$$\nabla_\alpha\nabla_\beta V^{\alpha\beta} = \tilde\nabla_i\tilde\nabla_k V^{ik} + \Big(s\frac{\partial}{\partial s}+4\Big)\tilde g_{ik}V^{ik} + \Big(2\frac{\partial}{\partial s}+\frac{8}{s}\Big)\tilde\nabla_i V^{0i} + \Big(\frac{\partial^2}{\partial s^2}+\frac{6}{s}\frac{\partial}{\partial s}+\frac{6}{s^2}\Big)V^{00} \quad (7.52)$$

which is valid identically for any symmetric $V^{\alpha\beta}$. That every index "0" on $V^{\cdots}$ is accompanied here by an extra power of $1/s$ (or $\partial/\partial s$) is a general rule. A similar calculation for a symmetrized fourth-rank tensor yields

$$\begin{aligned}
\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu V^{(\alpha\beta\mu\nu)} &= \tilde\nabla_i\tilde\nabla_k\tilde\nabla_m\tilde\nabla_n V^{(ikmn)} + \Big(6s\frac{\partial}{\partial s}+32\Big)\tilde g_{ik}\tilde\nabla_m\tilde\nabla_n V^{(ikmn)} \\
&+ \Big(3s^2\frac{\partial^2}{\partial s^2}+33s\frac{\partial}{\partial s}+72\Big)\tilde g_{ik}\tilde g_{mn}V^{(ikmn)} \\
&+ \text{terms in } \frac{1}{s}V^{0kmn},\ \frac{1}{s^2}V^{00mn},\ \frac{1}{s^3}V^{000n},\ \frac{1}{s^4}V^{0000}\ . \qquad (7.53)
\end{aligned}$$

In Eqs. (7.52) and (7.53) there figure the coordinate components of $V^{\alpha\ldots}$ i.e. the projections

$$V^{0\ldots} = \nabla_\alpha s\, V^{\alpha\ldots}\ \ ,\ \ V^{i\ldots} = \nabla_\alpha\gamma^i\, V^{\alpha\ldots}\ . \qquad (7.54)$$

The virtue of the introduced coordinate frame is in the fact that the set $\nabla_\alpha s$, $\nabla_\alpha\gamma^i$ is not a finite vector basis at i^+ as distinct from the coordinate basis of the Minkowski frame. From the transformation law (7.47) it follows that

$$\nabla_\alpha s\Big|_{i^+} = O(1)\ \ ,\ \ \nabla_\alpha\gamma^i\Big|_{i^+} = O\Big(\frac{1}{s}\Big)\ . \qquad (7.55)$$

Therefore, the projection of $V^{\alpha\cdots}$ in (7.54) that has n indices "i" has also the extra $1/s^n$ as compared to $V^{\alpha\ldots}$ in the Minkowski frame. With regard for this fact, all terms of an expression like (7.52) or (7.53) are, generally, of one and the same order at i^+. We have

$$\nabla_\alpha\nabla_\beta V^{\alpha\beta}\Big|_{i^+} = O\Big(\frac{1}{s^2}V\Big)\ \ ,\ \ \nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu V^{\alpha\beta\mu\nu}\Big|_{i^+} = O\Big(\frac{1}{s^4}V\Big)\ . \qquad (7.56)$$

This is precisely what is needed for the vertex functions (7.40)-(7.42) since inserted in (7.56) they all become $O(1/s)$. Specifically, the spatial components of all vertex functions in the frame (7.54) are

$$\hat V^{ik}_{\rm cross}\Big|_{i^+} = O\Big(\frac{1}{s}\Big)\ \ ,\ \ \hat V^{ik}_{\rm vect}\Big|_{i^+} = O\Big(\frac{1}{s}\Big)\ \ ,\ \ \hat V^{ikmn}_{\rm grav}\Big|_{i^+} = O\Big(\frac{1}{s}\Big)\ . \qquad (7.57)$$

The inference above obtains without using the conservation equation (7.44). Using this equation one can assert more, namely that all terms in (7.52) and (7.53) with at least

one index "0" on $V^{\cdots}$ can be discarded since they decrease by one power of $1/s$ faster than the "i" terms. Using also (7.57), one obtains for both $\hat{V}^{\alpha\beta}_{\rm cross}$ and $\hat{V}^{\alpha\beta}_{\rm vect}$ the expression

$$\nabla_\alpha\nabla_\beta V^{\alpha\beta}\Big|_{i^+} = \Big(\tilde\nabla_i\tilde\nabla_k + 3\tilde g_{ik}\Big)V^{ik} + O\Big(\frac{1}{s^2}\Big) \qquad (7.58)$$

and for $\hat{V}^{\alpha\beta\mu\nu}_{\rm grav}$ the expression

$$\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu V^{(\alpha\beta\mu\nu)}\Big|_{i^+} = \Big(\tilde\nabla_i\tilde\nabla_k\tilde\nabla_m\tilde\nabla_n + 26\tilde g_{ik}\tilde\nabla_m\tilde\nabla_n + 45\tilde g_{ik}\tilde g_{mn}\Big)V^{(ikmn)} + O\Big(\frac{1}{s^2}\Big). \quad (7.59)$$

These expressions are next subject to the integration over the parameters at i^+ with the measure in (6.45):

$$\int\limits_0^1 d\gamma\gamma^2\int d^2\mathcal{S}(\phi)\frac{1}{(1-\gamma^2)^{5/2}}(\ldots) = \int d^3\gamma\sqrt{\tilde g}\;\theta(1-\gamma)\;\frac{1}{\sqrt{1-\gamma^2}}(\ldots)\ . \qquad (7.60)$$

Therefore, one needs them only up to terms vanishing in this integral. Relation (7.50) makes it easy to integrate by parts in (7.60) or, equivalently, to bring expressions (7.58) and (7.59) to the following form:

$$\begin{aligned}\nabla_\alpha\nabla_\beta V^{\alpha\beta}\Big|_{i^+} &= 4\tilde g_{ik}V^{ik} + \sqrt{1-\gamma^2}\tilde\nabla_i Y^i\ , &(7.61)\\ \nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu V^{(\alpha\beta\mu\nu)}\Big|_{i^+} &= 72\tilde g_{ik}\tilde g_{mn}V^{(ikmn)} + \sqrt{1-\gamma^2}\tilde\nabla_i Z^i &(7.62)\end{aligned}$$

with some quantities Y^i and Z^i in the total derivative terms. It can be checked that these quantities vanish indeed at $\gamma=1$ which is the boundary of integration in (7.60).

One may now come back to the covariant form. With the metric in (7.48) one has

$$s^2\tilde g_{ik}V^{ik..} = (g_{\alpha\beta} + \nabla_\alpha s\nabla_\beta s)V^{\alpha\beta..}\ , \qquad (7.63)$$

and the projection $\nabla_\alpha s V^{\alpha\cdots}$ vanishes at i^+ by virtue of the conservation equation (7.44). Thus we obtain the following final results:

$$\begin{aligned}\nabla_\alpha\nabla_\beta V^{\alpha\beta}\Big|_{i^+} &= \frac{4}{s^2}g_{\alpha\beta}V^{\alpha\beta} + \text{ total derivatives}\ , &(7.64)\\ \nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu V^{(\alpha\beta\mu\nu)}\Big|_{i^+} &= \frac{72}{s^4}g_{\alpha\beta}g_{\mu\nu}V^{(\alpha\beta\mu\nu)} + \text{ total derivatives}\ . &(7.65)\end{aligned}$$

Substituting for $V^{\alpha\beta}$ and $V^{\alpha\beta\mu\nu}$ the expressions (7.40)-(7.42) we obtain the behaviours at i^+ of all non-scalar vertices (the total derivatives are omitted):

$$\nabla_\alpha\nabla_\beta\hat{V}^{\alpha\beta}_{\rm cross}\Big|_{i^+[\gamma,\phi,s\to\infty]} = -\frac{1}{12s}(1-\gamma^2)^{5/2}\int\limits_{-\infty}^{\infty}d\tau\Big(\frac{d^2}{d\tau^2}D^R\Big)\Big(\frac{d^2}{d\tau^2}\hat{D}^Q\Big)\ , \quad (7.66)$$

$$\nabla_\alpha \nabla_\beta \hat{V}^{\alpha\beta}_{\rm vect}\Big|_{i^+[\gamma,\phi,s\to\infty]} = \frac{1}{12s}(1-\gamma^2)^{3/2}\int\limits_{-\infty}^{\infty} d\tau g_{\alpha\beta}\Big(\frac{d}{d\tau}\hat{D}^\alpha\Big)\Big(\frac{d}{d\tau}\hat{D}^\beta\Big) \ , \qquad (7.67)$$

$$\nabla_\alpha \nabla_\beta \nabla_\mu \nabla_\nu \hat{V}^{\alpha\beta\mu\nu}_{\rm grav}\Big|_{i^+[\gamma,\phi,s\to\infty]} = -\frac{\hat{1}}{120s}(1-\gamma^2)^{5/2}\int\limits_{-\infty}^{\infty} d\tau (g_{\alpha\mu}g_{\beta\nu}+\frac{1}{2}g_{\alpha\beta}g_{\mu\nu}) \times \Big(\frac{d^2}{d\tau^2}D^{\alpha\beta}\Big)\Big(\frac{d^2}{d\tau^2}D^{\mu\nu}\Big) \ . \qquad (7.68)$$

With the behaviour of $\hat{V}_{\rm scalar}$ in (7.39) included, we have the complete list. All the behaviours are of the form (6.42) i.e. proportional to $1/s$, and for all but one the proportionality coefficient contains the factor $(1-\gamma^2)^{5/2}$ as required in Eq. (6.46). The exception is the vector vertex (7.67) in which instead of $(1-\gamma^2)^{5/2}$ we have $(1-\gamma^2)^{3/2}$. This is a consequence of the lack of convergence at $\mathcal{I}^+$. That the results are correct can be checked by comparing the asymptotic expressions above with the behaviours in the future of $\mathcal{I}^+$, Eqs. (6.51)-(6.54). The correspondence (1.29) between the limits i^+ and $\mathcal{I}^+$ holds in all cases.

An alternative way of handling the non-scalar vertices is considered in Appendix B.

The vertices at $\mathcal{T}^+$.

Since we assume that all timelike geodesics are infinitely extendable to the future, the behaviour (7.1) holds at $\mathcal{T}^+$ as well. The parameter $\sqrt{-2q}$ is then to be shifted by s like in Eq. (7.30), and, as a result, there will appear integrals over the hypersurfaces orthogonal to the geodesics coming to $\mathcal{T}^+$. These integrals analogous to the moments will be stationary at late time like in Eq. (7.35) since their only ingredients will be the metric and the stationary source. It follows that, in terms of the proper time along the world lines filling the tube, the behaviours of the vertex operators at $\mathcal{T}^+$ are the same as at i^+:

$$F(m,n)J_1J_2\Big|_{\mathcal{T}^+} \propto s^{m+n-3} \ , \quad s\to\infty \ . \qquad (7.69)$$

Since we assume that there is no future horizon, the proper time of an observer in the tube is analytically related to the external time u (normalized in (1.14)). Hence

$$F(m,n)J_1J_2\Big|_{\mathcal{T}^+} \propto u^{m+n-3} \ , \quad u\to\infty \ . \qquad (7.70)$$

Eq. (7.70) signifies that the vertex terms in the scalar I have one and the same power of growth at $\mathcal{T}^+$ and i^+ [19] whereas, for making a contribution to the real vacuum energy, the growth at $\mathcal{T}^+$ should be by three powers of u faster (Sec.6). Thus we infer that the only contribution to the total vacuum energy comes from the vertex functions at i^+.

[19] For $\hat{V}_{\rm scalar}$ this follows directly from (7.70). For the non-scalar vertices the overall derivatives in the expressions like $\nabla_\alpha\nabla_\beta\hat{V}^{\alpha\beta}_{\rm vect}$, etc. have at $\mathcal{T}^+$ the same effect as at i^+. This is clear from the treatment of these derivatives in Appendix B.

8 Creation of particles and radiation of waves

The energy of the vacuum particle production.

The result of consideration in the previous section is that formula (6.45) can indeed be used, and $W(\gamma,\phi)$ in this formula is the sum of expressions (7.39),(7.66),(7.67) and (7.68) with the factors $1/s$ detached:

$$\begin{aligned} W(\gamma,\phi) \;=\; & -\frac{1}{4}(1-\gamma^2)^{5/2}\int\limits_{-\infty}^{\infty} d\tau\,\mathrm{tr}\Big[\Big(\frac{d^2}{d\tau^2}\hat{D}^Q\Big)\Big(\frac{d^2}{d\tau^2}\hat{D}^Q\Big)+\frac{1}{3}\Big(\frac{d^2}{d\tau^2}D^R\Big)\Big(\frac{d^2}{d\tau^2}\hat{D}^Q\Big) \\ & -\frac{1}{3}\frac{1}{(1-\gamma^2)}g_{\alpha\beta}\Big(\frac{d}{d\tau}\hat{D}^\alpha\Big)\Big(\frac{d}{d\tau}\hat{D}^\beta\Big) \\ & +\frac{\hat{1}}{30}(g_{\alpha\mu}g_{\beta\nu}+\frac{1}{2}g_{\alpha\beta}g_{\mu\nu})\Big(\frac{d^2}{d\tau^2}D^{\alpha\beta}\Big)\Big(\frac{d^2}{d\tau^2}D^{\mu\nu}\Big)\Big]\;. \end{aligned} \tag{8.1}$$

By (1.12)

$$\hat{D}^Q=\hat{D}-\frac{\hat{1}}{6}D^R \tag{8.2}$$

where $\hat{D}$ is the radiation moment (7.19) of the potential matrix $\hat{P}$. One may notice that the coefficients of the first two terms in (8.1) are precisely such that the cross contribution $\hat{P}\times R$ cancels. This cancellation (although nontrivial) is a consequence of adding $-\frac{1}{6}R\hat{1}$ to the potential in Eq. (1.2) and is the only manifestation of the conformal properties of the effective action [31]. Otherwise, these properties play no role in the present calculation as the reader can see. [20]

Using also that

$$D^R=-g_{\mu\nu}D^{\mu\nu} \tag{8.3}$$

(cf. Eq. (4.37)) and inserting (8.1) in (6.45) we obtain the following *final result* for the

[20] There is an unceasing controversy in connection with the old paper [32] about the significance of the trace anomaly in dimensions higher than two. The calculation in the present paper may serve as a commentary to the following conversation that took place between two persons:

N: The anomaly is a window through which we can see ...

V: ... the back yard.

quantity (1.37) in terms of the radiation moments (7.19)-(7.21):

$$\begin{aligned} M(-\infty) - M(\infty) &= \frac{1}{(4\pi)^2}\int\limits_0^1 d\gamma\gamma^2 \int\limits_{-\infty}^{\infty} d\tau \int d^2\mathcal{S}(\phi)\ \mathrm{tr}\Big[\Big(\frac{d^2}{d\tau^2}\hat{D}\Big)^2 \\ &\qquad - \frac{1}{3}\frac{1}{(1-\gamma^2)} g_{\alpha\beta}\Big(\frac{d}{d\tau}\hat{D}^\alpha\Big)\Big(\frac{d}{d\tau}\hat{D}^\beta\Big) \\ &\qquad + \frac{1}{30}\hat{1}(g_{\mu\alpha}g_{\nu\beta} - \frac{1}{3}g_{\mu\nu}g_{\alpha\beta})\Big(\frac{d^2}{d\tau^2}D^{\alpha\beta}\Big)\Big(\frac{d^2}{d\tau^2}D^{\mu\nu}\Big)\Big]\ . \end{aligned} \tag{8.4}$$

There is a pole at $\gamma = 1$ in the term with the vector moment. However, under the limitation (6.57):

$$\mathrm{tr}\Big[g_{\alpha\beta}\Big(\frac{d}{d\tau}\hat{D}^\alpha\Big)\Big(\frac{d}{d\tau}\hat{D}^\beta\Big)\Big]\Big|_{\gamma=1} = 0 \tag{8.5}$$

the pole cancels. As shown below, this limitation is a condition that the vector connection field contains no outgoing wave.

Expression (8.4) is the total energy of the particles produced from the vacuum by external fields. It is useful to realize that the integral over γ in (8.4) is none other than the integral over the energies of the outgoing particles. The vacuum radiation is thus obtained along with its spectrum.

Positivity.

Owing to the conservation laws for the moments, all contractions with the metric in (8.4) are positive definite. In particular, by (1.23) and (7.24),

$$\begin{aligned} &(g_{\mu\alpha}g_{\nu\beta} - \frac{1}{3}g_{\mu\nu}g_{\alpha\beta})\Big(\frac{d^2}{d\tau^2}D^{\alpha\beta}\Big)\Big(\frac{d^2}{d\tau^2}D^{\mu\nu}\Big) = \frac{1}{2}\Big|\frac{d^2}{d\tau^2}D^{\alpha\beta}m_\alpha m_\beta\Big|^2 \\ &+ 2(1-\gamma^2)\Big|\frac{d^2}{d\tau^2}D^{\alpha\beta}\nabla_\alpha r\, m_\beta\Big|^2 + \frac{1}{6}\Big(\frac{d^2}{d\tau^2}D^{\alpha\beta}m_\alpha m^*_\beta - 2(1-\gamma^2)\frac{d^2}{d\tau^2}D^{\alpha\beta}\nabla_\alpha r\nabla_\beta r\Big)^2 \end{aligned} \tag{8.6}$$

which proves the positivity of the gravitational-field contribution in (8.4).

The positivity of the matrix contributions follows from assuming the self-adjointness of the operator of small disturbances of the quantum field [21]. There should exist a

[21]For simplicity, let this field be boson and non-gauge. In the general case the calculation will include diagonalizing and squaring operators, and combining loops [24]. Eq. (8.4) gives the contribution of a generic loop.

symmetric and nondegenerate matrix ω_{AB} such that the operator

$$\omega_{AC}H^C_B \tag{8.7}$$

with H^A_B in (1.1) is the Hessian of an action. The self-adjointness then requires that i) the operator (8.7) be symmetric, and ii) the matrix ω_{AB} be positive definite. The latter is a condition of the absence of ghosts. The symmetry of the operator (8.7) implies that i) ω_{AB} behaves like a metric with respect to the covariant differentiation:

$$\nabla_\mu \omega_{AB} = 0 \ \ , \tag{8.8}$$

and ii) ω_{AB} converts the potential matrix into a symmetric form:

$$P^C_A\omega_{CB} - P^C_B\omega_{CA} = 0 \ \ . \tag{8.9}$$

From (8.8) it follows that

$$[\nabla_\mu, \nabla_\nu]\omega_{AB} = -\mathcal{R}^C_{A\mu\nu}\omega_{CB} - \mathcal{R}^C_{B\mu\nu}\omega_{CA} = 0 \ \ , \tag{8.10}$$

i.e. ω_{AB} converts the commutator curvature into an antisymmetric form. Using (8.8) once again one obtains the antisymmetry relation for the source of the commutator curvature in (1.7):

$$J^{C\alpha}_A\omega_{CB} + J^{C\alpha}_B\omega_{CA} = 0 \ \ , \ \ J^{A\alpha}_B = \hat{J}^\alpha \ \ . \tag{8.11}$$

At this point it is important that the integrals with matrices contain the propagators of parallel transport for the matrix indices. The explicit forms of Eqs. (7.19) and (7.20) are

$$D^A_B = \frac{1}{4\pi}\int d\bar{x}\bar{g}^{1/2}\delta\Big(T(x,\bar{x}) - \tau\Big) g^A{}_{\bar{A}} g_B{}^{\bar{B}}\, \bar{P}^{\bar{A}}_{\bar{B}}\Big|_{x\to i^+} \ \ , \tag{8.12}$$

$$D^{A\alpha}_B = \frac{1}{4\pi}\int d\bar{x}\bar{g}^{1/2}\delta\Big(T(x,\bar{x}) - \tau\Big) g^A{}_{\bar{A}} g_B{}^{\bar{B}} g^\alpha{}_{\bar{\alpha}} \bar{J}^{\bar{A}\bar{\alpha}}_{\bar{B}}\Big|_{x\to i^+} \ \ . \tag{8.13}$$

By (8.8), the law of parallel transport for ω_{AB} is

$$\omega_{AB}\, g^A{}_{\bar{A}} g^B{}_{\bar{B}} = \bar{\omega}_{\bar{A}\bar{B}} \ \ . \tag{8.14}$$

Using this law it is not difficult to prove that the symmetries of the sources in (8.12) and (8.13) imply the symmetries of the moments:

$$D^C_A\,\omega_{CB} = D^C_B\,\omega_{CA} \ , \tag{8.15}$$

$$D^{C\alpha}_A\,\omega_{CB} = -D^{C\alpha}_B\,\omega_{CA} \ . \tag{8.16}$$

The latter symmetries remain unchanged under a differentiation with respect to τ and a contraction of (8.16) with any vector.

Let $\hat{\Gamma}^{+}$ and $\hat{\Gamma}^{-}$ be any matrices possessing the properties

$$\Gamma^{\pm C}_{\ A}\,\omega_{CB} = \pm\Gamma^{\pm C}_{\ B}\,\omega_{CA}\ . \tag{8.17}$$

The positive-definite metric ω_{AB} can be expanded over an orthonormal basis:

$$\omega_{AB} = h_A(M)h_B(N)\delta(M,N)\ \ ,\ \ h_A(M)h^A(N) = \delta(M,N)\ \ ,\ \ h^A(M) = \omega^{-1AB}h_B(M) \tag{8.18}$$

with $\delta(M,N)$ the Kronecker symbol. Denoting

$$\Gamma^{\pm}(M,N) = h^A(M)\Gamma^{\pm B}_{\ A}h_B(N) \tag{8.19}$$

one has

$$\mathrm{tr}\Big(\hat{\Gamma}^{\pm}\Big)^2 = \Gamma^{\pm A}_{\ B}\Gamma^{\pm B}_{\ A} = \pm\Gamma^{\pm A}_{\ B}\omega_{AC}\Gamma^{\pm C}_{\ D}\omega^{-1DB} = \pm\sum_{N,M}\Big(\Gamma^{\pm}(M,N)\Big)^2\ . \tag{8.20}$$

Thus, identifying $\hat{\Gamma}^{\pm}$ with

$$\hat{\Gamma}^{+} = \frac{d^2}{d\tau^2}\hat{D}\ \ ,\ \ \hat{\Gamma}^{-} = \frac{d}{d\tau}\hat{D}^{\alpha}k_{\alpha} \tag{8.21}$$

where k_α is an arbitrary vector, we obtain

$$\mathrm{tr}\Big(\frac{d^2}{d\tau^2}\hat{D}\Big)^2 \geq 0\quad ,\quad \mathrm{tr}\Big(\frac{d}{d\tau}\hat{D}^{\alpha}k_{\alpha}\Big)^2 \leq 0\ . \tag{8.22}$$

The first of these inequalities proves the positivity of the potential contribution in (8.4). Finally, by (1.23) and (7.23),

$$g_{\alpha\beta}\Big(\frac{d}{d\tau}\hat{D}^{\alpha}\Big)\Big(\frac{d}{d\tau}\hat{D}^{\beta}\Big) = \Big(\frac{d}{d\tau}\hat{D}^{\alpha}m_{\alpha}\Big)\Big(\frac{d}{d\tau}\hat{D}^{\beta}m^{*}_{\beta}\Big) + (1-\gamma^2)\Big(\frac{d}{d\tau}\hat{D}^{\alpha}\nabla_{\alpha}r\Big)^2\ . \tag{8.23}$$

Hence, by the second inequality in (8.22),

$$\mathrm{tr}\Big[g_{\alpha\beta}\Big(\frac{d}{d\tau}\hat{D}^{\alpha}\Big)\Big(\frac{d}{d\tau}\hat{D}^{\beta}\Big)\Big] \leq 0 \tag{8.24}$$

which proves the positivity of the commutator-curvature contribution in (8.4).

The matrix ω_{AB} itself does not figure in any of the final expressions (8.22)-(8.24). Only its existence is important.

Radiation of waves.

Generally, the flux of the non-coherent radiation caused by pair creation in the vacuum is only one term of the mass-loss formula (1.35). To clear up the meaning of condition (8.5) and to complete the discussion of the radiation moments, we shall consider also the other terms which in the field-theoretic case stand for a radiation of waves (not necessarily classical [23]).

By analogy with the electromagnetic field we may assume that the contribution of the vector connection field to $T^{\mu\nu}_{\rm source}$ in (1.32) is of the form [22]

$$T^{\mu\nu}_{\rm source} = -\frac{1}{4\pi}{\rm tr}\Big(g_{\alpha\beta}\hat{\mathcal{R}}^{\mu\alpha}\hat{\mathcal{R}}^{\nu\beta} - \frac{1}{4}g^{\mu\nu}\hat{\mathcal{R}}^{\alpha\beta}\hat{\mathcal{R}}_{\alpha\beta}\Big) + \text{a contribution of the potential } \hat{P} \quad (8.25)$$

where the change of the overall sign as compared to the case where $\hat{\mathcal{R}}^{\mu\nu}$ is literally replaced by the Maxwell tensor is owing to the antisymmetry (8.10). The density of the flux of $T^{\mu\nu}_{\rm source}$ through $\mathcal{I}^+$ is then

$$\begin{aligned} \frac{1}{4}r^2 T^{\mu\nu}_{\rm source}\nabla_\mu v\nabla_\nu v\Big|_{\mathcal{I}^+} &= -\frac{1}{4\pi}{\rm tr}\Big(\frac{1}{4}r^2\nabla_\mu v\,\hat{\mathcal{R}}^{\mu\alpha}g_{\alpha\beta}\hat{\mathcal{R}}^{\nu\beta}\nabla_\nu v\Big)\Big|_{\mathcal{I}^+} \\ &= -\frac{1}{4\pi}{\rm tr}\Big[\Big(\frac{1}{2}r\nabla_\mu v\,\hat{\mathcal{R}}^{\mu\alpha}m_\alpha\Big)\Big(\frac{1}{2}r\nabla_\nu v\,\hat{\mathcal{R}}^{\nu\beta}m^*_\beta\Big)\Big]_{\mathcal{I}^+} \end{aligned} \quad (8.26)$$

where the last form is obtained by inserting expression (1.21) for $g_{\alpha\beta}$ and using the antisymmetry of $\hat{\mathcal{R}}^{\alpha\beta}$. If we define the complex news function [23] for the waves of the vector connection field

$$\frac{\partial}{\partial u}\mathbf{C}_{\rm vect}(u,\phi) = -\frac{1}{2}\nabla_\alpha v\, m_\beta\Big(r\hat{\mathcal{R}}^{\alpha\beta}\Big)\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} \quad , \quad (8.27)$$

then, according to (8.26) and (1.35), the quantity

$$-\frac{1}{4\pi}\int d^2\mathcal{S}(\phi){\rm tr}\Big|\frac{\partial}{\partial u}\mathbf{C}_{\rm vect}(u,\phi)\Big|^2 \quad (8.28)$$

is the energy flux of the outgoing radiation of this field.

[22] Outside the support of the source of $\hat{\mathcal{R}}^{\mu\nu}$.

[23] This term was introduced in [26,27] for the gravitational waves but the electromagnetic waves can be discussed along the same lines. Two real components of the news function are the initial data for the two independent degrees of freedom of a radiation field counted per point of $\mathcal{I}$.

Assuming that there is no incoming radiation, we may use the Jacobi identities to express $\hat{\mathcal{R}}^{\alpha\beta}$ through its source as in Eq. (A.17) of Appendix A. Using also (4.17) we obtain to lowest order in $\Re$

$$\left(r\hat{\mathcal{R}}^{\alpha\beta}\right)\Big|_{\mathcal{I}^+[\tau,\phi,r\to\infty]} = \nabla^\alpha u\,\frac{d}{d\tau}\hat{D}_{\mathbf{1}}^{\beta}(\tau,\phi) - \nabla^\beta u\,\frac{d}{d\tau}\hat{D}_{\mathbf{1}}^{\alpha}(\tau,\phi) \quad . \tag{8.29}$$

Hence

$$\frac{\partial}{\partial\tau}\mathbf{C}_{\mathrm{vect}}(\tau,\phi) = m_\alpha\frac{d}{d\tau}\hat{D}_{\mathbf{1}}^{\alpha}(\tau,\phi) \quad , \tag{8.30}$$

and

$$\Big|\frac{\partial}{\partial\tau}\mathbf{C}_{\mathrm{vect}}(\tau,\phi)\Big|^2 = \Big|\frac{d}{d\tau}\hat{D}_{\mathbf{1}}^{\alpha}m_\alpha\Big|^2 = (\delta_{ik} - n_i n_k)\Big(\frac{d}{d\tau}\hat{D}_{\mathbf{1}}^{i}\Big)\Big(\frac{d}{d\tau}\hat{D}_{\mathbf{1}}^{k}\Big) \tag{8.31}$$

where, in the last form, $\hat{D}_{\mathbf{1}}^{i}$ are the spatial components of $\hat{D}_{\mathbf{1}}^{\alpha}$ in the Minkowski frame, and use is made of Eq. (1.28). Using the basis decomposition (1.21) for the metric and the conservation law (4.42) for the moment, expression (8.31) can be written in the covariant form

$$\Big|\frac{\partial}{\partial\tau}\mathbf{C}_{\mathrm{vect}}(\tau,\phi)\Big|^2 = g_{\alpha\beta}\Big(\frac{d}{d\tau}\hat{D}_{\mathbf{1}}^{\alpha}\Big)\Big(\frac{d}{d\tau}\hat{D}_{\mathbf{1}}^{\beta}\Big) \quad . \tag{8.32}$$

The total energy of the waves of the vector field emitted for the whole history is, therefore,

$$-\frac{1}{4\pi}\int\limits_{-\infty}^{\infty} d\tau \int d^2\mathcal{S}(\phi)\mathrm{tr}\Big[g_{\alpha\beta}\Big(\frac{d}{d\tau}\hat{D}^{\alpha}\Big)\Big(\frac{d}{d\tau}\hat{D}^{\beta}\Big)\Big]_{\gamma=1} \quad . \tag{8.33}$$

It is now seen that condition (8.5) can be written in the form

$$\mathrm{tr}\Big|\frac{\partial}{\partial\tau}\mathbf{C}_{\mathrm{vect}}(\tau,\phi)\Big|^2 \equiv 0 \quad , \tag{8.34}$$

and its meaning is that the source of the vector field radiates no waves.

Only the $\gamma = 1$ moments $D_{\mathbf{1}}$ appear in classical radiation theory as is clear from Eq. (4.17). However, even in this theory the concept of boosted (γ - dependent) moment D is useful since it helps to understand the nature of the multipole expansion. In the case of a nonrelativistic source, γ is the only parameter that contains the velocity of light since γ is the velocity of a particle at i^+ per unit velocity of light. Therefore, in the nonrelativistic approximation, D at $\gamma = 1$ may be calculated as an expansion at $\gamma = 0$:

$$\hat{D}_{\mathbf{1}}^{\alpha} = \hat{D}^{\alpha}\Big|_{\gamma=0} + \frac{d}{d\gamma}\hat{D}^{\alpha}\Big|_{\gamma=0} + \dots \quad . \tag{8.35}$$

This expansion gives rise to the multipole moments. Since at $\gamma = 0$ the dependence on the direction at i^+ should disappear, the expansion in γ is also an expansion in the direction vector n^i. For the spatial components of $\hat{D}^\alpha$ in the Minkowski frame one may write quite generally

$$\hat{D}^i\Big|_{\gamma=0} = \frac{d}{d\tau}\,\hat{d}^i \quad , \tag{8.36}$$

$$\frac{d}{d\gamma}\hat{D}^i\Big|_{\gamma=0} = n_k\Big(\frac{1}{6}\frac{d^2}{d\tau^2}\,\hat{d}^{(ki)} + \frac{d}{d\tau}\,\hat{d}^{[ki]}\Big) + \text{a term} \propto n^i \quad , \quad \hat{d}^{(ki)}\delta_{ki} = 0 \tag{8.37}$$

and so on. Here $d^i, d^{(ki)}, d^{[ki]}$ are the dipole, quadrupole and magnetic moments, and the term $\propto n^i$ drops out of (8.31). The appearance of the time derivatives in expressions (8.36) and (8.37) is owing to the fact that the multipole moments are coefficients in the expansion of the $\mathcal{D}$ in Eq. (7.28) rather than the D. For the radiation of a nonrelativistic source one obtains the standard textbook result

$$\frac{1}{4\pi}\int d^2\mathcal{S}(\phi)\,\Big|\frac{\partial}{\partial\tau}\mathbf{C}_{\text{vect}}(\tau,\phi)\Big|^2 = \frac{2}{3}\Big(\frac{d^2}{d\tau^2}\,d^i\Big)^2 + \frac{1}{180}\Big(\frac{d^3}{d\tau^3}\,d^{(ik)}\Big)^2 + \frac{1}{3}\Big(\frac{d^2}{d\tau^2}\,d^{[ik]}\Big)^2 + \ldots \quad . \tag{8.38}$$

The (differentiated) news function for the outgoing gravitational waves is [24]

$$\frac{\partial^2}{\partial u^2}\mathbf{C}_{\text{grav}}(u,\phi) = -\frac{1}{8}\nabla_\alpha v\nabla_\mu v\, m_\beta m_\nu\Big(rR^{\alpha\beta\mu\nu}\Big)\Big|_{\mathcal{I}^+[u,\phi,r\to\infty]} \quad . \tag{8.39}$$

Assuming again that there are no incoming waves and solving the Bianchi identities to express $R^{\alpha\beta\mu\nu}$ through its source [15,23] we obtain to lowest order in $\Re$

$$\Big(rR^{\alpha\beta\mu\nu}\Big)\Big|_{\mathcal{I}^+[\tau,\phi,r\to\infty]} = -4\nabla^{[\mu}u\nabla^{<\alpha}u\,\frac{d^2}{d\tau^2}\,D_{\mathbf{1}}^{\nu]\beta>}(\tau,\phi) \tag{8.40}$$

where both types of brackets [] and <> denote antisymmetrization of the respective indices. Hence

$$\frac{\partial^2}{\partial\tau^2}\mathbf{C}_{\text{grav}}(\tau,\phi) = \frac{1}{2}m_\alpha m_\beta\frac{d^2}{d\tau^2}D_{\mathbf{1}}^{\alpha\beta}(\tau,\phi) \quad . \tag{8.41}$$

With the initial condition (5.41) one may integrate this equation to obtain the news function

$$\frac{\partial}{\partial\tau}\mathbf{C}_{\text{grav}}(\tau,\phi) = \frac{1}{2}m_\alpha m_\beta\frac{d}{d\tau}D_{\mathbf{1}}^{\alpha\beta}(\tau,\phi) \quad . \tag{8.42}$$

[24] See, e.g., [23] and the original references [26,27].

Thus

$$\left|\frac{\partial}{\partial\tau}\mathbf{C}_{\rm grav}(\tau,\phi)\right|^2 = \frac{1}{4}\left|\frac{d}{d\tau}D_{\mathbf{1}}^{\alpha\beta}m_\alpha m_\beta\right|^2 = \tag{8.43}$$
$$= \frac{1}{2}\Big[(\delta_{im}-n_in_m)(\delta_{kn}-n_kn_n) - \frac{1}{2}(\delta_{ik}-n_in_k)(\delta_{mn}-n_mn_n)\Big]\Big(\frac{d}{d\tau}D_{\mathbf{1}}^{ik}\Big)\Big(\frac{d}{d\tau}D_{\mathbf{1}}^{mn}\Big)$$

where, in the last form, $D_{\mathbf{1}}^{ik}$ are the spatial components of $D_{\mathbf{1}}^{\alpha\beta}$ in the Minkowski frame. Bringing this expression to the covariant form yields the result

$$\left|\frac{\partial}{\partial\tau}\mathbf{C}_{\rm grav}(\tau,\phi)\right|^2 = \frac{1}{2}(g_{\alpha\mu}g_{\beta\nu} - \frac{1}{2}g_{\alpha\beta}g_{\mu\nu})\Big(\frac{d}{d\tau}D_{\mathbf{1}}^{\alpha\beta}\Big)\Big(\frac{d}{d\tau}D_{\mathbf{1}}^{\mu\nu}\Big) \ . \tag{8.44}$$

Therefore, according to Eq. (1.35), the total energy of the gravitational waves emitted for the whole history is

$$\frac{1}{4\pi}\int\limits_{-\infty}^{\infty} d\tau \int d^2\mathcal{S}(\phi)\frac{1}{2}(g_{\alpha\mu}g_{\beta\nu} - \frac{1}{2}g_{\alpha\beta}g_{\mu\nu})\Big(\frac{d}{d\tau}D^{\alpha\beta}\Big)\Big(\frac{d}{d\tau}D^{\mu\nu}\Big)\Big|_{\gamma=1} \ . \tag{8.45}$$

For a nonrelativistic source,

$$D^{\alpha\beta} = D^{\alpha\beta}\Big|_{\gamma=0} + O(\gamma) \ , \tag{8.46}$$

and

$$D^{ik}\Big|_{\gamma=0} = \frac{1}{3}\frac{d^2}{d\tau^2}\, d^{ik} + \text{a term } \propto \delta^{ik} \ , \quad d^{ik}\delta_{ik} = 0 \tag{8.47}$$

where d^{ik} is the quadrupole moment, and the term $\propto \delta^{ik}$ drops out of (8.43). Here again, d^{ik} is the coefficient in the expansion of the $\mathcal{D}$ in Eq. (7.29) rather than the D. Therefore, in (8.47) there appears the second time derivative. For the radiation of the gravitational waves by a nonrelativistic source Eqs. (8.43) and (8.47) yield the textbook result

$$\frac{1}{4\pi}\int d^2\mathcal{S}(\phi)\left|\frac{\partial}{\partial\tau}\mathbf{C}_{\rm grav}(\tau,\phi)\right|^2 = \frac{1}{45}\Big(\frac{d^3}{d\tau^3}\, d^{ik}\Big)^2 + \ldots \ . \tag{8.48}$$

Compare Eqs. (8.33) and (8.45) with (8.4). The similarity is striking. As a result of the derivation above, the quantum problem of particle creation becomes almost the same thing as the classical problem of radiation of waves. The whole difference is that, in the quantum problem, there figure the boosted moments D and, instead of setting $\gamma = 1$, one is *to integrate over* γ. In the nonrelativistic case even this difference disappears since all moments will be expanded at $\gamma = 0$, and the integral over γ will be removed trivially.

Note also that in the case of the vector field both the classical and quantum radiation effects are determined by the second-order time derivative of the moment (in terms of the $\mathcal{D}$, Eq. (7.27)). In the case of the gravitational field there appears one more distinction between the classical and quantum contributions to the radiation energy. Namely, the classical radiation is determined by the third-order and quantum by the fourth-order time derivative of the respective moment $\mathcal{D}$. This is a consequence of the dimension of the coupling constant in the quantum loop.

9 Specializations and examples

Spherical symmetry.

The fact that the energy of vacuum radiation is expressed through the moments with γ different from unity has an important consequence. Inspecting Eqs. (8.6), (8.23) and (8.43) one can see that at $\gamma = 1$ there survive only the projections of the moments on the 2-sphere $\mathcal{S}$ (the transverse projections) whereas at $\gamma \neq 1$ there are also projections on ∇r i.e. on the direction of motion of the partice at i^+ (the longitudinal projections). For a spherically symmetric source, the vector moment can have no projection on the sphere, and the projection of the tensor moment on the sphere can only be proportional to the metric on the sphere in which case the orthogonality $(m, m) = 0$ comes into effect. It follows that spherically symmetric sources cannot emit waves but can produce particles from the vacuum. This makes spherical symmetry an interesting case for studying the effect of vacuum radiation. In addition, the limitation (8.5) is in this case fulfilled automatically and so we can study the particle production by vector fields.

For spherically symmetric sources we have from (8.23) and (8.6)

$$g_{\alpha\beta}\Big(\frac{d}{d\tau}\hat{D}^\alpha\Big)\Big(\frac{d}{d\tau}\hat{D}^\beta\Big) = (1-\gamma^2)\Big(\frac{d}{d\tau}\hat{D}^\alpha\nabla_\alpha r\Big)^2 \ , \tag{9.1}$$

$$(g_{\mu\alpha}g_{\nu\beta} - \frac{1}{3}g_{\mu\nu}g_{\alpha\beta})\Big(\frac{d^2}{d\tau^2}D^{\alpha\beta}\Big)\Big(\frac{d^2}{d\tau^2}D^{\mu\nu}\Big) = \frac{1}{6}\Big(\frac{d^2}{d\tau^2}D^R + 3(1-\gamma^2)\frac{d^2}{d\tau^2}D^{\alpha\beta}\nabla_\alpha r\nabla_\beta r\Big)^2 \ . \tag{9.2}$$

Since these quantities are spherically symmetric scalars, they do not depend on the angles at i^+. Therefore, the integral over ϕ in (8.4) removes trivially, and we obtain

$$\begin{aligned} M(-\infty) - M(\infty) \ = \ & \frac{1}{4\pi}\int\limits_0^1 d\gamma\gamma^2 \int\limits_{-\infty}^{\infty} d\tau \ \mathrm{tr}\Big[\Big(\frac{d^2}{d\tau^2}\hat{D}\Big)^2 - \frac{1}{3}\Big(\frac{d}{d\tau}\hat{D}^\alpha\nabla_\alpha r\Big)^2 \\ & + \frac{1}{180}\hat{1}\Big(\frac{d^2}{d\tau^2}D^R + 3(1-\gamma^2)\frac{d^2}{d\tau^2}D^{\alpha\beta}\nabla_\alpha r\nabla_\beta r\Big)^2\Big] \end{aligned} \tag{9.3}$$

where there is no more pole in γ including in the term with the vector moment.

The longitudinal projections of the moments are conveniently obtained as follows. One differentiates the law of parallel transport (7.14) with respect to γ. This yields

$$\nabla_\mu r(x)\, g^\mu{}_{\bar\mu}(x,\bar x)\Big|_{x\to i^+[\gamma,\phi]} = -\bar\nabla_{\bar\mu}\frac{\partial}{\partial\gamma}T_{\gamma\phi}(\bar x) + (\nabla_\mu t(x) - \gamma\nabla_\mu r(x))\frac{\partial}{\partial\gamma}\, g^\mu{}_{\bar\mu}(x,\bar x)\Big|_{x\in i^+[\gamma,\phi]} \quad . \tag{9.4}$$

The derivative

$$\frac{\partial}{\partial\gamma}\Big(\, g^\mu{}_{\bar\mu}(x,\bar x)\Big|_{x\in i^+[\gamma,\phi]}\Big) = O[\Re] \tag{9.5}$$

can be calculated as a result of parallel transport of a vector along the closed contour consisting of two geodesics which emanate from $\bar x$ and come to i^+ with boosts γ and $\gamma+\delta\gamma$. With the contribution of (9.5) neglected , one obtains from (7.20) and (7.21)

$$\nabla_\alpha r\hat D^\alpha = -\frac{1}{4\pi}\int d\bar x\bar g^{1/2}\delta(T_{\gamma\phi}(\bar x)-\tau)\Big(\bar\nabla_{\bar\mu}\frac{\partial}{\partial\gamma}\bar T_{\gamma\phi}\Big)\hat J^{\bar\mu}(\bar x) + O[\Re^2] \ , \tag{9.6}$$

$$\nabla_\alpha r\nabla_\beta r D^{\alpha\beta} = \frac{1}{4\pi}\int d\bar x\bar g^{1/2}\delta(T_{\gamma\phi}(\bar x)-\tau)\Big(\bar\nabla_{\bar\mu}\frac{\partial}{\partial\gamma}\bar T_{\gamma\phi}\Big)\Big(\bar\nabla_{\bar\nu}\frac{\partial}{\partial\gamma}\bar T_{\gamma\phi}\Big)J^{\bar\mu\bar\nu}(\bar x) + O[\Re^2] \ . \tag{9.7}$$

The metric of a spherically symmetric spacetime is generally of the form

$$ds^2 = d\Gamma^2 + r^2\Big|_\Gamma d\mathcal{S}^2 \ , \ d\Gamma^2 = g_{AB}(y)dy^Ady^B \ , \ d\mathcal{S}^2 = g_{ab}(\phi)d\phi^ad\phi^b \tag{9.8}$$

where Γ is some 2-dimensional Lorentzian space, $\mathcal{S}$ is the unit 2-sphere, and $r = r(y)$ is a function on Γ. The line $r = 0$ is a boundary in Γ. We shall denote g^Γ the determinant of $g_{AB}(y)$ and ∇_A the derivative in Γ. The world function on a spherically symmetric spacetime is a regular function of the following arguments:

$$\sigma(x,\bar x) = \sigma(y,\bar y,\cos\omega) \ , \quad \omega = \omega(\phi,\bar\phi) \tag{9.9}$$

where $\frac{1}{2}\omega^2(\phi,\bar\phi)$ is the world function on the unit 2-sphere, and an expression for $\cos\omega$ is given in (4.16). Hence

$$T_{\gamma\phi}(\bar x) = T_\gamma(\bar y,\cos\omega) \ , \quad \omega = \omega(\phi,\bar\phi) \tag{9.10}$$

and Eq. (7.4) takes the form

$$\bar g^{AB}\bar\nabla_A T_\gamma\bar\nabla_B T_\gamma + \frac{1}{\bar r^2}(1-\cos^2\omega)\Big(\frac{d}{d\cos\omega}T_\gamma\Big)^2 = -(1-\gamma^2) \ . \tag{9.11}$$

Denote

$$T_\gamma^{\pm} = T_\gamma(y,\pm 1) \ . \tag{9.12}$$

By (9.11), the integral curves of each of the vector fields

$$N_+^A(y) = \frac{1}{\sqrt{1-\gamma^2}} \nabla^A T_\gamma^+ \quad , \quad N_-^A(y) = \frac{1}{\sqrt{1-\gamma^2}} \nabla^A T_\gamma^- \tag{9.13}$$

are timelike geodesics in Γ. With fixed γ, the geodesics generated by N_+ make a 1-parameter congruence foliating Γ. The geodesics generated by N_- make another such congruence. The geodesics from the two congruences may be joined pairwise, a pair consisting of the two geodesics that hit one and the same point of the boundary $r = 0$. A world line in Γ consisting of two geodesics joined at $r = 0$ is the mapping on Γ of a 4-dimensional radial geodesic (the broken line in Fig.7). This line is generated by N_+ in the future from $r = 0$ and by N_- in the past from $r = 0$.

The spacelike lines orthogonal to the geodesics generated by N_+ also make a 1-parameter congruence foliating Γ. The spacelike lines orthogonal to the geodesics generated by N_- make one more such congruence. These congruences are respectively

$$T_\gamma^+ = \tau \quad , \quad T_\gamma^- = \tau \tag{9.14}$$

with τ the parameter of the congruence. By (9.11),

$$\frac{d}{d\cos\omega}\Big(T_\gamma(\bar{y}, \cos\omega)\Big|_{r(\bar{y})=0}\Big) = 0 \quad . \tag{9.15}$$

It follows that the spacelike lines (9.14) join pairwise at $r = 0$ as shown in Fig.7. The lines $T_\gamma^+ = \tau$ and $T_\gamma^- = \tau$ with one and the same τ hit one and the same point of the boundary. *The hyperplane $T_{\gamma\phi}(x) = \tau$ maps on the interior domain between these two lines.* However, the boundary of this domain, i.e. the line

$$\Big(T_\gamma^+ = \tau\Big) \cup \Big(T_\gamma^- = \tau\Big) \tag{9.16}$$

itself corresponds to no 4-dimensional original. Only in flat spacetime is this line a mapping of a spacelike geodesic.

In two limiting cases, $\gamma = 1$ and $\gamma = 0$, there come about coincidences. At $\gamma = 1$, the line $T_\gamma^+ = \tau$ coincides with a geodesic generated by N_+ , and the line $T_\gamma^- = \tau$ coincides with a geodesic generated by N_- . Both geodesics are then null, and their union is the mapping on Γ of a 4-dimensional radial light ray. At $\gamma = 0$, the line $T_\gamma^+ = \tau$ coincides

with the line $T_\gamma^{\,-} = \tau$. In this case the mapping of the hyperplane on Γ degenerates into a line.

When all external fields are spherically symmetric, there appear only moments of the sources $\bar{X}$ that either do not depend on the angles $\bar{\phi}$ or depend on them through $\omega(\phi, \bar{\phi})$ where ϕ are the angles at i^+. The moment of any such source can be calculated as follows:

$$D = \frac{1}{4\pi} \int d\bar{x} \bar{g}^{1/2} \delta\Big(T_{\gamma\phi}(\bar{x}) - \tau\Big) \bar{X} = \frac{1}{2} \int d^2\bar{y} \sqrt{-\bar{g}^\Gamma}\, \bar{r}^2 \int\limits_{-1}^{1} d(\cos\omega) \delta\Big(T_\gamma(\bar{y}, \cos\omega) - \tau\Big) \bar{X} \ . \tag{9.17}$$

Here the equation of the integration hyperplane is to be solved with respect to $\cos\omega$. Let $\cos\omega = f$ be the solution. Then the moment is obtained as the integral in Γ

$$D = \frac{1}{2} \int d^2\bar{y} \sqrt{-\bar{g}^\Gamma}\, \theta\Big(T_\gamma^{\,-} - \tau\Big) \theta\Big(\tau - T_\gamma^{\,+}\Big)\, \bar{r}^2 \left|\frac{dT_\gamma}{d\cos\omega}\right|^{-1} \bar{X}\Big|_{\cos\omega = f} \tag{9.18}$$

in which the integration domain is bounded by the lines $T_\gamma^{\,-} = \tau$ and $T_\gamma^{\,+} = \tau$. This calculation will be carried out below for flat spacetime, and here it will only be noted that the Jacobian appearing in (9.18) is

$$\left|\frac{dT_\gamma}{d\cos\omega}\right| = \frac{\sqrt{1-\gamma^2}}{\sin\omega} \Big|L(\bar{y}, \omega, \gamma)\Big| \ , \quad \omega = \omega(\phi, \bar{\phi}) \tag{9.19}$$

where $L(\bar{y}, \omega, \gamma)$ is the angular momentum of the geodesic that, when traced backwards in time, comes from $i^+[\gamma, \phi]$ to the point $(\bar{y}, \bar{\phi})$ of a compact domain. This follows from the geodesic equation (7.7) whose angle component takes the form

$$\frac{d\omega}{ds} = -\frac{1}{r^2} \frac{1}{\sqrt{1-\gamma^2}} \frac{d}{d\omega} T_\gamma(y, \cos\omega) \ . \tag{9.20}$$

Electrically charged shell expanding in the self field.

It is not the purpose of the present paper to discuss applications of the result (8.4) but to give a simple example we shall consider the particle production by an electrically charged spherical shell expanding in the self field. Below, e, m, and $\mathcal{E}$ are respectively the shell's charge, rest mass, and energy in excess of the rest energy.

In terms of time t orthogonal to r , $(\nabla t, \nabla r) \equiv 0$, a spherically symmetric electromagnetic field is completely determined by a single arbitrary function $e(r,t)$ which is the charge contained on the hypersurface $t =$ const. inside the 2-sphere of area $4\pi r^2$:

$$e(r,t) = \int d\bar{x}\bar{g}^{1/2}\delta(\bar{t}-t)\theta(r-\bar{r})\bar{\nabla}_{\bar{\mu}}\bar{t}\,\bar{j}^{\bar{\mu}} \quad , \tag{9.21}$$

$$\nabla_\nu F^{\mu\nu} = 4\pi j^\mu \quad . \tag{9.22}$$

Here $F^{\mu\nu}$ is the Maxwell tensor, and j^μ is the electromagnetic current. The choice of $e(r,t)$ is limited only by the condition of regularity of the field at $r = 0$, $e(0,t) = 0$, and the condition that the support of j^μ belongs to a spacetime tube. Let $r = \rho(t)$ be the equation of the boundary of the tube. Then

$$e(r,t)\Big|_{r>\rho(t)} = e = \text{const.} \tag{9.23}$$

where e is the full charge. In flat spacetime, with the normalization of time $(\nabla t)^2 = -1$, the general solution of the conservation equation is

$$4\pi j^\mu = -(\nabla^\mu t)\frac{1}{r^2}\frac{\partial e(r,t)}{\partial r} - (\nabla^\mu r)\frac{1}{r^2}\frac{\partial e(r,t)}{\partial t} \quad , \tag{9.24}$$

and the general solution of the Maxwell equations is

$$F^{\mu\nu} = E^\mu \nabla^\nu t - E^\nu \nabla^\mu t \quad , \quad E^\mu = \frac{e(r,t)}{r^2}\nabla^\mu r \tag{9.25}$$

where only an electric field is present.

In the case where the support of the charge is a thin shell, we have

$$e(r,t) = e\,\theta(r-\rho(t)) \tag{9.26}$$

and, by (9.24),

$$4\pi(\nabla_\mu t)j^\mu = \frac{e}{\rho^2(t)}\delta(r-\rho(t)) \quad , \tag{9.27}$$

$$4\pi(\nabla_\mu r)j^\mu = -\Big(\frac{d}{dt}\frac{e}{\rho(t)}\Big)\delta(r-\rho(t)) \tag{9.28}$$

where $r = \rho(t)$ is a law of motion of the shell. In this case there is only a static Coulomb field outside the shell; inside the field vanishes. However, this simplicity is only apparent.

Because the shell should expand in the self field, this field is nonstationary, and the law of expansion is not very simple [25]:

$$\frac{mc^2}{\sqrt{1-\frac{1}{c^2}\Big(\frac{d\rho}{dt}\Big)^2}}+\frac{1}{2}\frac{e^2}{\rho}=mc^2+\mathcal{E}\ \ . \tag{9.29}$$

The shell can be contracted as much as one wants by raising its energy but for given energy its minimum radius is

$$r_{\min}=\frac{e^2}{2\mathcal{E}}\ \ . \tag{9.30}$$

Set free, it expands monotonically from $r=r_{\min}$ to $r=\infty$ with increasing speed.

We shall consider two limiting cases, the nonrelativistic shell

$$\frac{d\rho}{dt}=\sqrt{\frac{1}{m}\Big(2\mathcal{E}-\frac{e^2}{\rho}\Big)}\ \ ,\ \ \mathcal{E}<<mc^2 \tag{9.31}$$

and the ultrarelativistic shell

$$\rho=\frac{e^2}{2\mathcal{E}}\Big(1-\frac{mc^2}{\mathcal{E}}\Big)+\sqrt{c^2(t-t_{\rm start})^2+\Big(\frac{e^2}{2\mathcal{E}}\frac{mc^2}{\mathcal{E}}\Big)^2}\ \ ,\ \ t_{\rm start}=\text{const.}\ \ ,\ \ \mathcal{E}>>mc^2\ . \tag{9.32}$$

It is important that in the latter case the shell will remain timelike. The timelike ultrarelativistic shell is always slow near $r=r_{\min}$ and only at $r\to\infty$ it approaches the speed of light.

Particle production by a spherically symmetric electromagnetic field.

The expanding spherical shell emits no electromagnetic waves but, as we show below, it excites the vacuum of charged fields, and the vacuum emits the quanta of these fields.

The coupling of an external electromagnetic field A_μ to the charged vacuum fields will be introduced through the following form of the covariant derivative in (1.2):

$$\nabla_\mu=\partial_\mu\hat{1}+qA_\mu\hat{\Omega}\quad ,\quad {\rm tr}\,\hat{\Omega}^2<0 \tag{9.33}$$

[25] The appearance of 1/2 in the Coulomb energy is owing to the fact that the force exerted on the surface charge is determined by one half of the sum of the electric fields on both sides of the surface [33].

where q is the charge of the vacuum particles, $\hat{\Omega}$ is a numerical matrix, and the negativity of the trace of $\hat{\Omega}^2$ is a corollary of the self-adjointness of the equation of the quantum field (see Sec.8).

For the commutator curvature and its source, Eq. (9.33) yields the expressions

$$\hat{\mathcal{R}}^{\mu\nu} = q\,\hat{\Omega}F^{\mu\nu} \quad , \quad \hat{J}^{\mu} = 4\pi q\,\hat{\Omega}\,j^{\mu} \tag{9.34}$$

in terms of the Maxwell tensor and the electromagnetic current. It is convenient to factor the coupling parameters out of the longitudinal projection of the vector moment:

$$\nabla_{\alpha} r\,\hat{D}^{\alpha} \equiv q\,\hat{\Omega}D_{||} \quad . \tag{9.35}$$

Then for $D_{||}$ we obtain from (9.6)

$$D_{||} = -\int d\bar{x}\bar{g}^{1/2}\delta\Big(T_{\gamma\phi}(\bar{x}) - \tau\Big)\Big(\bar{\nabla}_{\bar{\mu}}\frac{\partial}{\partial\gamma}\bar{T}_{\gamma\phi}\Big)\bar{j}^{\bar{\mu}} \quad . \tag{9.36}$$

With the metric of the Lorentzian section in (9.8)

$$d\Gamma^2 = -dt^2 + dr^2 \tag{9.37}$$

one may use expression (7.17) for the hyperplane:

$$T_{\gamma\phi}(\bar{x}) = \bar{t} - \bar{r}\gamma\cos\omega(\phi,\bar{\phi}) \quad . \tag{9.38}$$

Hence

$$T_{\gamma}^{\pm} = \bar{t} \mp \gamma\bar{r} \quad , \tag{9.39}$$

and the lines (9.14) are spacelike geodesics (Fig. 7). With a spherically symmetric current j^{μ} we obtain

$$\Big(\bar{\nabla}_{\bar{\mu}}\frac{\partial}{\partial\gamma}\bar{T}_{\gamma\phi}\Big)\bar{j}^{\bar{\mu}} = -\cos\omega(\bar{\nabla}_{\bar{\mu}}\bar{r})\bar{j}^{\bar{\mu}} \quad , \tag{9.40}$$

and Eq. (9.36) takes the form

$$D_{||} = \int d^2\mathcal{S}(\bar{\phi})\int\limits_0^{\infty} d\bar{r}\,\bar{r}^2\int\limits_{-\infty}^{\infty} d\bar{t}\;\delta(\bar{t} - \tau - \bar{r}\gamma\cos\omega)\cos\omega(\bar{\nabla}_{\bar{\mu}}\bar{r})\bar{j}^{\bar{\mu}} \quad . \tag{9.41}$$

The integration over the angles in this expression brings the longitudinal moment to its final form

$$D_{||} = \frac{2\pi}{\gamma^2}\int\limits_{-\infty}^{\infty} dt\int\limits_0^{\infty} dr\;\theta(t - \tau + r\gamma)\theta(\tau - t + r\gamma)\,(t-\tau)(\nabla_{\mu}r)j^{\mu} \tag{9.42}$$

which is a specialization of (9.18). Here the longitudinal projection of a spherically symmetric current is to be inserted from (9.24):

$$(\nabla_\mu r) j^\mu = -\frac{1}{4\pi r^2}\frac{\partial e(r,t)}{\partial t} \quad . \tag{9.43}$$

In terms of $D_{||}$ the vacuum energy released by a spherically symmetric electromagnetic field is, by (9.3) and (9.35),

$$M(-\infty) - M(\infty) = \frac{q^2}{12\pi}\Big(-\mathrm{tr}\ \hat\Omega^2\Big) \int\limits_0^1 d\gamma\gamma^2 \int\limits_{-\infty}^{\infty} d\tau \Big(\frac{d}{d\tau} D_{||}\Big)^2 \quad . \tag{9.44}$$

Radiation of the nonrelativistic shell.

The longitudinal moment of the charged spherical shell is obtained by inserting expression (9.28) in (9.42):

$$D_{||} = -\frac{e}{2\gamma^2}\int\limits_{t_-}^{t_+} dt\ (t-\tau)\ \frac{d}{dt}\frac{1}{\rho(t)} \quad . \tag{9.45}$$

Here t_+ and t_- are solutions of the following equations:

$$t_+ - \gamma\rho(t_+) = \tau \quad , \quad t_- + \gamma\rho(t_-) = \tau \quad . \tag{9.46}$$

These are the time instants at which the world line of the shell intersects the lines (9.14). The moment (9.45) is an integral along the world line of the shell between the points t_- and t_+ as shown in Fig.7. Integrating by parts in (9.45), and using Eq. (9.46) we obtain

$$D_{||} = \frac{e}{2\gamma^2}\Big(\int\limits_{t_-}^{t_+} dt \frac{1}{\rho(t)} - 2\gamma\Big) \quad . \tag{9.47}$$

For the nonrelativistic shell, the moment can be calculated by expanding it at $\gamma = 0$. Since, by (9.46),

$$t_+ = \tau + O(\gamma) \quad , \quad t_- = \tau + O(\gamma) \quad , \tag{9.48}$$

this amounts to expanding $\rho(t)$ at $t = \tau$. A straightforward calculation using only Eq. (9.46) yields

$$\int\limits_{t_-}^{t_+} dt \frac{1}{\rho(t)} = 2\gamma + \frac{1}{3}\gamma^3 \frac{d^2}{d\tau^2}\rho^2(\tau) + O(\gamma^4) \tag{9.49}$$

whence

$$D_{||} = \frac{e\gamma}{6}\frac{d^2}{d\tau^2}\rho^2(\tau) + O(\gamma^2) \ . \tag{9.50}$$

Inserting this expression in (9.44) one obtains the result

$$M(-\infty) - M(\infty) = \frac{q^2 e^2}{180 \times 12\pi}(-\mathrm{tr}\ \hat{\Omega}^2) \int\limits_{-\infty}^{\infty} d\tau \Big(\frac{d^3}{d\tau^3}\rho^2(\tau)\Big)^2 \tag{9.51}$$

which holds for the nonrelativistic shell irrespectively of the specific form of the law $\rho(\tau)$.

It is now important to know the entire history of the shell. We shall assume that before some time instant $t = t_{\mathrm{start}}$ the shell was kept at the point of maximum contraction $r = r_{\min}$ and next was let go. Beginning with $t = t_{\mathrm{start}}$ it was expanding unboundedly in the self field. This world line is shown in Fig.7.

Since

$$\frac{d\rho(t)}{dt}\bigg|_{t<t_{\mathrm{start}}} = 0 \ , \tag{9.52}$$

we have

$$\int\limits_{-\infty}^{\infty} dt \Big(\frac{d^3}{dt^3}\rho^2(t)\Big)^2 = \int\limits_{t_{\mathrm{start}}}^{\infty} dt \Big[(\rho^2)'''\Big]^2 = \int\limits_{r_{\min}}^{\infty} \frac{d\rho}{\rho'}\Big[(\rho^2)'''\Big]^2 \tag{9.53}$$

where the primes denote the derivatives of $\rho(t)$, $r_{\min}$ is given in (9.30), and the replacement of the integration variable $t \to \rho(t)$ is to be made with the law $\rho(t)$ in (9.31). It is convenient to write

$$(\rho^2)''' = 6\rho'\rho'' + 2\rho\rho'\frac{d}{d\rho}(\rho'') \tag{9.54}$$

and use the law of motion (9.31):

$$\rho'' = \frac{e^2}{2m}\frac{1}{\rho^2} \ , \quad \rho' = \sqrt{\frac{2\mathcal{E}}{m} - \frac{e^2}{m\rho}} \ . \tag{9.55}$$

One obtains

$$\int\limits_{r_{\min}}^{\infty} \frac{d\rho}{\rho'}\Big[(\rho^2)'''\Big]^2 = \frac{(2\mathcal{E})^{7/2}}{e^2 m^{5/2}} \int\limits_{1}^{\infty} dx \frac{\sqrt{x-1}}{x^{9/2}} \ , \tag{9.56}$$

and

$$\int\limits_{1}^{\infty} dx \frac{\sqrt{x-1}}{x^{9/2}} = 2\,\frac{2}{3}\,\frac{2}{5}\,\frac{2}{7} \ . \tag{9.57}$$

Finally, restoring $\hbar$ and c, we infer that for the whole time of expansion the nonrelativistic shell looses the energy

$$M(-\infty) - M(\infty) = \frac{(-\mathrm{tr}\ \hat{\Omega}^2)}{81 \times 25 \times 7\pi}\,\frac{q^2}{\hbar c}\Big(\frac{2\mathcal{E}}{mc^2}\Big)^{5/2} 2\mathcal{E} \ , \quad \mathcal{E} << mc^2 \ . \tag{9.58}$$

Radiation of the ultrarelativistic shell.

Coming back to the exact expression (9.47) for the longitudinal moment, and using (9.46) we obtain

$$\frac{dt_+}{d\tau} = \frac{1}{1-\gamma\frac{d\rho}{dt}\Big|_{t_+}} \quad , \quad \frac{dt_-}{d\tau} = \frac{1}{1+\gamma\frac{d\rho}{dt}\Big|_{t_-}} \tag{9.59}$$

and

$$\frac{d}{d\tau}D_{||} = \frac{e}{2\gamma^2}\left(\frac{1}{\rho\Big(1-\gamma\frac{d\rho}{dt}\Big)\Big|_{t_+}} - \frac{1}{\rho\Big(1+\gamma\frac{d\rho}{dt}\Big)\Big|_{t_-}}\right) \tag{9.60}$$

where the notation assumes that the substitution $t = t_+$ or $t = t_-$ is to be made.

Since the law $\rho(t)$ is different for $t < t_{\rm start}$ and $t > t_{\rm start}$, the range of integration over τ should be divided into three intervals,

$$\int\limits_{-\infty}^{\infty} d\tau\Big(\frac{d}{d\tau}D_{||}\Big)^2 = \Big(\int\limits_{-\infty}^{\tau_1} d\tau + \int\limits_{\tau_1}^{\tau_2} d\tau + \int\limits_{\tau_2}^{\infty} d\tau\Big)\Big(\frac{d}{d\tau}D_{||}\Big)^2 \ , \tag{9.61}$$

according to the location of the points t_+ and t_-:

$$\begin{array}{rcl} -\infty < \tau < \tau_1 & : & t_- < t_{\rm start} \ , \ t_+ < t_{\rm start} \\ \tau_1 < \tau < \tau_2 & : & t_- < t_{\rm start} \ , \ t_+ > t_{\rm start} \\ \tau_2 < \tau < \infty & : & t_- > t_{\rm start} \ , \ t_+ > t_{\rm start} \ . \end{array}$$

The motion of the points t_+ and t_- along the world line of the shell as τ increases can be traces in Fig. 7. We have

$$\int\limits_{-\infty}^{\tau_1} d\tau\Big(\frac{d}{d\tau}D_{||}\Big)^2 = 0 \ , \tag{9.62}$$

$$\int\limits_{\tau_1}^{\tau_2} d\tau\Big(\frac{d}{d\tau}D_{||}\Big)^2 = \frac{e^2}{4\gamma^4}\int\limits_{\tau_1}^{\tau_2} d\tau\left[\frac{1}{\rho\Big(1-\gamma\frac{d\rho}{dt}\Big)\Big|_{t_+}} - \frac{1}{r_{\rm min}}\right]^2 \ , \tag{9.63}$$

$$\int\limits_{\tau_2}^{\infty} d\tau\Big(\frac{d}{d\tau}D_{||}\Big)^2 = \frac{e^2}{4\gamma^4}\int\limits_{\tau_2}^{\infty} d\tau\left[\frac{1}{\rho\Big(1-\gamma\frac{d\rho}{dt}\Big)\Big|_{t_+}} - \frac{1}{\rho\Big(1+\gamma\frac{d\rho}{dt}\Big)\Big|_{t_-}}\right]^2 \ , \tag{9.64}$$

and, from (9.46),

$$\tau_1 = t_{\rm start} - \gamma r_{\rm min} \ , \ \tau_2 = t_{\rm start} + \gamma r_{\rm min} \ . \tag{9.65}$$

In (9.63) and (9.64) the law $\rho(t)$ pertains already to the expansion stage.

Consider a shell that moves all the time with the speed of light. For such a shell, assuming that it passes through $r = r_{\rm min}$ at $t = t_{\rm start}$, the solutions of Eqs. (9.46) are

$$\rho(t_\pm) = \frac{\tau - t_{\rm start} + r_{\rm min}}{1 \mp \gamma} \quad , \quad \frac{d\rho(t)}{dt} \equiv 1 \quad , \tag{9.66}$$

and expression (9.60) vanishes identically. Hence we infer that *a null shell creates no real energy from the vacuum.* [26] Now consider a shell that before $t = t_{\rm start}$ is at rest and then suddenly starts expanding with the speed of light. This shell creates an infinite amount of energy owing to the jump of its velocity at $t = t_{\rm start}$. Indeed, in this case the term (9.64) vanishes

$$\int\limits_{\tau_2}^{\infty} d\tau \Big(\frac{d}{d\tau} D_{||}\Big)^2 \equiv 0 \tag{9.67}$$

but the term (9.63) doesn't:

$$\int\limits_{\tau_1}^{\tau_2} d\tau \Big(\frac{d}{d\tau} D_{||}\Big)^2 = \frac{\mathcal{E}}{2\gamma^4}\Big(\frac{1}{1-\gamma} - \frac{1}{1+\gamma} + 2\log\frac{1-\gamma}{1+\gamma} + 2\gamma\Big) \quad . \tag{9.68}$$

The latter expression behaves like $O(1/\gamma^2)$ at $\gamma = 0$ which in view of the presence of γ^2 in the measure in (9.44) is a regular behaviour. However, at $\gamma = 1$ expression (9.68) has a pole, and the total energy (9.44) diverges:

$$\Big(M(-\infty) - M(\infty)\Big) \propto -\mathcal{E}\log(1-\gamma)\Big|_{\gamma=1} \quad . \tag{9.69}$$

The pole comes from the lower limit in (9.68) at which the regime changes from static to null. These considerations suggest that, in the case of the timelike ultrarelativistic shell, the effect may come only from a neighbourhood of $r = r_{\rm min}$ and it should be finite owing to the continuity of the shell's velocity. It is clear in advance that the divergent term (9.69) will be regularized with the regularization parameter $mc^2/\mathcal{E}$, and the regularized term will be of order $\mathcal{E}\log(\mathcal{E}/mc^2)$. Since this term grows as $mc^2/\mathcal{E} \to 0$, it will make a dominant contribution to the energy of the particle production.

Indeed, for the timelike shell, Eq. (9.67) gets replaced with

$$\int\limits_{\tau_2}^{\infty} d\tau \Big(\frac{d}{d\tau} D_{||}\Big)^2 \to 0 \quad , \quad \frac{mc^2}{\mathcal{E}} \to 0 \tag{9.70}$$

[26] This is not the case for a gravitationally charged shell contracting in the self field.

while the term (9.63) can be expressed directly through the law of motion (9.32) by making the change $\tau \to t_+(\tau)$ of the integration variable and inserting the Jacobian from (9.59):

$$\int_{\tau_1}^{\tau_2} d\tau\Big(\frac{d}{d\tau}D_{||}\Big)^2 = \frac{e^2}{4\gamma^4}\int_{t_+(\tau_1)}^{t_+(\tau_2)} dt\Big(1-\gamma\frac{d\rho(t)}{dt}\Big)\Bigg[\frac{1}{\rho(t)\Big(1-\gamma\frac{d\rho(t)}{dt}\Big)} - \frac{1}{r_{\min}}\Bigg]^2 \tag{9.71}$$
$$= \frac{e^2}{4\gamma^4}\int_{r_{\min}}^{\rho(t_+(\tau_2))} \frac{d\rho}{\rho'}\Big[\frac{1}{\rho^2(1-\gamma\rho')} - \frac{2}{\rho r_{\min}} + \frac{(1-\gamma\rho')}{r_{\min}^2}\Big] \ .$$

Here the approximation $\rho' = 1$ can be used in all terms except in the denominator $(1-\gamma\rho')$ where the deceleration of the shell near $r = r_{\min}$ cannot be neglected. Specifically, the upper limit in (9.71) can be calculated in the approximation $\rho' = 1$ using (9.66) and (9.65):

$$\rho(t_+(\tau_2)) = \frac{1+\gamma}{1-\gamma}r_{\min} \ , \quad \frac{mc^2}{\mathcal{E}} \to 0 \ . \tag{9.72}$$

However, for the insertion in the denominator $(1-\gamma\rho')$, the law of motion (9.32) should be approximated better:

$$\rho' = \frac{\rho - r_{\min}}{\sqrt{(\rho - r_{\min})^2 + (r_{\min}\frac{mc^2}{\mathcal{E}})^2}} \ . \tag{9.73}$$

In this way, introducing the integration variable

$$x = \frac{\rho - r_{\min}}{r_{\min}} \ , \tag{9.74}$$

we obtain for $mc^2/\mathcal{E} \to 0$

$$\int_{-\infty}^{\infty} d\tau\Big(\frac{d}{d\tau}D_{||}\Big)^2 = \frac{\mathcal{E}}{2\gamma^4}\Big\{\frac{1}{1-\gamma} - \frac{1}{1+\gamma} + 2\log\frac{1-\gamma}{1+\gamma} + 2\gamma \tag{9.75}$$
$$-\frac{\gamma}{1-\gamma}\Big(\frac{mc^2}{\mathcal{E}}\Big)^2\int_0^{2\gamma/(1-\gamma)} dx\frac{1}{(1+x)^2[(1-\gamma^2)x^2 + (mc^2/\mathcal{E})^2]}\Big\} \ .$$

This expression has already a regular behaviour at both $\gamma = 0$ and $\gamma = 1$ (cf. Eq. (9.68)).

Upon doing the integral in (9.75) and next the integral over γ one is to retain the senior terms at the limit $mc^2/\mathcal{E} \to 0$. These are seen to be the senior terms of (9.75) at the limit

$$(1-\gamma) \to 0 \ , \quad \frac{mc^2}{\mathcal{E}} \to 0 \ , \quad (1-\gamma)\Big(\frac{\mathcal{E}}{mc^2}\Big)^2 = \text{finite} \ . \tag{9.76}$$

At this limit expression (9.75) becomes

$$\int_{-\infty}^{\infty} d\tau\Big(\frac{d}{d\tau}D_{||}\Big)^2 = \mathcal{E}\int_0^{\infty} dx\frac{x^2}{(1+x)^2[2(1-\gamma)x^2+(mc^2/\mathcal{E})^2]} \quad . \tag{9.77}$$

Hence, interchanging the integrations over x and γ , we obtain

$$\int_0^1 d\gamma\gamma^2\int_{-\infty}^{\infty} d\tau\Big(\frac{d}{d\tau}D_{||}\Big)^2 = \mathcal{E}\Big(\log\frac{\mathcal{E}}{mc^2}\Big)\int_0^{\infty} dx\frac{1}{(1+x)^2} + O(1) \quad , \quad \frac{mc^2}{\mathcal{E}}\to 0 \tag{9.78}$$

where the contribution comes from the end point $\gamma = 1$.

Thus we infer that for the whole time of expansion the ultrarelativistic shell looses the energy

$$M(-\infty) - M(\infty) = \frac{(-\mathrm{tr}\,\hat{\Omega}^2)}{12\pi}\frac{q^2}{\hbar c}\Big(\log\frac{\mathcal{E}}{mc^2}\Big)\mathcal{E} \quad , \quad \mathcal{E} >> mc^2 \quad . \tag{9.79}$$

This is already interesting since, at a sufficiently large $\mathcal{E}$, the shell radiates more energy than it has. Accounting for the backreaction of radiation becomes necessary.

Both quantities (9.58) and (9.79) are independent of the charge e of the shell and remain nonvanishing at the limit $e = 0$. This paradox is a consequence of the property of classical theory that the charge cannot be switched off gradually. The backreaction of the vacuum may change this result since the expectation-value equations contain a dimensionless parameter e/q . Below we show that the range $e/q \lesssim 1$ is within the limits of validity of the approximations made.

Validity of the approximations.

The calculational scheme presented uses two types of expansion. The effective action is expanded in the number of loops, and within each loop order it is expanded over the basis of nonlocal invariants like in Eq. (2.2) [27] . The limits of validity of the latter expansion are presently to be considered. These limits are best seen from a comparison of the technique of nonlocal form factors with the Schwinger-DeWitt technique [30].

[27] In the axiomatic approach of Refs. [19-22] the expansion in loops is avoided, and expansion (2.2) is considered as an ansatz for the full action.

As shown in [34], the nonlocal form factors emerge as a result of a partial summation of the local Schwinger-DeWitt series for the effective action. At a given order in the dimensional coupling constants this series can symbolically be written down as follows:

$$S_{\rm vac} = {\rm const.} \int dx g^{1/2} \sum_p \frac{1}{\mu^{2p}} \sum_{n=1}^{p} (\nabla)^{2p-2n} \Big(\Re\Big)^n \tag{9.80}$$

where μ is the mass of the vacuum particles. The nonlocal expansion of the effective action is obtained by summing all terms in (9.80) with a given power of $\Re$ and any number of derivatives. Within each order in μ^2 one neglects $\Re^{n+1}$ as compared to $\nabla\nabla\Re^n$. Therefore, the condition of validity of the resultant nonlocal expansion is generally of the form [34,19]

$$\Re^2 << \nabla\nabla\Re \ . \tag{9.81}$$

The partially summed series admits the limit $\mu \to 0$ [34,14-17] and at this limit takes the form (2.2). There is no problem of principle in generalizing the nonlocal expansion to the case of massive vacuum particles but the neglect of the mass is correct if

$$\mu^2\Re << \nabla\nabla\Re \ . \tag{9.82}$$

The approximation in which Eqs. (9.81) and (9.82) hold may be qualified as the high-frequency approximation meaning the frequency of the external (or mean) field.

Not all limitations implied in (9.81) and (9.82) should necessarily be fulfilled since not all terms neglected in the partial summation of the series (9.80) may be important for the problem in question. Thus, the effect of particle creation is caused by the external field's variability in time, and its spatial inhomogeneity is unimportant. Let l be the characteristic spatial scale of the source of the external field and ν be its characteristic frequency. Then, for the problem of particle creation, the important limitation stemming from (9.82) is

$$\mu c^2 << \hbar\nu \ . \tag{9.83}$$

Under this condition the results of the technique of nonlocal form factors are valid for massive vacuum particles as well.

When specializing to the problem of particle creation by an external electric field one is to consider only the mixed components of the commutator curvature $\hat{\mathcal{R}}_{\mu\nu}$. For these

components condition (9.81) yields

$$\hat{\mathcal{R}}_{0i} << \frac{\nu}{l} \quad , \tag{9.84}$$

and, by (9.34),

$$\hat{\mathcal{R}}_{0i} = \text{const.}\ \frac{q}{\hbar} E \tag{9.85}$$

where E is the characteristic strength of the external electric field:

$$E = \frac{e}{l^2} \quad . \tag{9.86}$$

Thus, for this problem, the condition of validity of the nonlocal expansion is

$$\frac{qe}{l} << \hbar\nu \tag{9.87}$$

where the quantity on the left-hand side is the Coulomb energy of the vacuum particle in the external electric field. Condition (9.87) is to be combined with the condition of validity of the one-loop approximation:

$$\frac{q^2}{\hbar c} << 1 \quad . \tag{9.88}$$

For the nonrelativistic shell, l and ν are unrelated:

$$l = \frac{e^2}{\mathcal{E}} \quad , \quad \nu = \frac{\mathcal{E}^{3/2}}{m^{1/2} e^2} \quad , \quad \mathcal{E} << mc^2 \quad . \tag{9.89}$$

In this case the energy of the shell is clutched by the inequalities (9.87) and (9.89) in the interval

$$\frac{q^2}{\hbar c}\, \frac{e}{q} << \sqrt{\frac{\mathcal{E}}{mc^2}} << 1 \quad . \tag{9.90}$$

Owing to condition (9.88) this still leaves room. It is only important that the ratio e/q be not too large:

$$\frac{q^2}{\hbar c}\, \frac{e}{q} << 1 \quad . \tag{9.91}$$

The weaker the coupling of the quantum field, the bigger charge of the external field can be considered. Thus, for the electron-positron vacuum, $e = 10q$ is admissible but $e = 137q$ is not.

For the ultrarelativistic sources we have $\nu l = c$, and the inequality (9.87) takes again the form (9.91). Thus condition (9.91) is universal but the relativistic case is better suited for the approximations made since no upper bound on the energy $\mathcal{E}$ emerges.

Concluding remarks

This section will not be long since it may be called "reminiscences of the future" with the exception of the following point.

As remarked in Sec.2, Eqs. (2.38)-(2.45) pertain to the case where $\hat{\mathcal{R}}_{\alpha\beta}$ and $\hat{P}$ are metric independent but *the final result* (8.4) *is valid for arbitrary local* $\hat{\mathcal{R}}_{\alpha\beta}$ *and* $\hat{P}$. An inspection of the table of asymptotic behaviours in [18] shows that, in the contributions of the variational derivatives $\delta\hat{\mathcal{R}}_{\alpha\beta}/\delta g_{\mu\nu}$ and $\delta\hat{P}/\delta g_{\mu\nu}$ to $T^{\mu\nu}_{\rm vac}\Big|_{\mathcal{I}^+}$, the highest exponents m and n of the vertex operators $F(m,n)$ are by one smaller than in (2.40)-(2.43). The contributions of these variational derivatives are, therefore, *pure quantum noise*. This remarkable fact gives the result (8.4) the status of a generating expression applicable to fields of any spin [24].

The remaining limitations imposed on the external fields require bigger efforts for their removal.

The limitation (8.5) signals that the theory contains another effect: the vacuum screening or amplification of the electromagnetic waves emitted by a source. This is equivalent to an *observable* renormalization of all multipole moments. The emergence of the limitation on the vector field is connected with the fact that, in the case of this field, the nonlocal vacuum polarization and the local charge renormalization are of one and the same order by dimension. The separation of the respective terms in the effective action is quite subtle (this issue is discussed for QED in [35]). Hence it is clear why no limitation like (8.5) emerges in the case of the gravitational field. It is also clear what should be done. Since there appear nonlocal sources that behave at $\mathcal{I}^+$ like $O(1/r^2)$ rather than $O(1/r^3)$, two amendments are needed. First, terms $O(\Box^0)$ of the form factors can no more be discarded. They should be extracted from the exact form factors [17] and added to (2.36). Second, expression (3.14) for the kernel of $\log(-\Box)$ is no more valid and should be replaced by the exact expression [21,22]. It can be expected that, after introducing these amendments, the difficulty with the convergence at $\mathcal{I}^+$ will be removed along with

the limitation (8.5), and the result will be the following expression for the density of the radiation flux from the vector source:

$$
-\frac{1}{4\pi}Zg_{\alpha\beta}\Big(\frac{d}{d\tau}\hat{D}^{\alpha}\Big)\Big(\frac{d}{d\tau}\hat{D}^{\beta}\Big)\Big|_{\gamma=1}
$$
$$
-\frac{1}{3(4\pi)^2}\int\limits_0^1 d\gamma\gamma^2\frac{1}{(1-\gamma^2)}\Big[g_{\alpha\beta}\Big(\frac{d}{d\tau}\hat{D}^{\alpha}\Big)\Big(\frac{d}{d\tau}\hat{D}^{\beta}\Big)-g_{\alpha\beta}\Big(\frac{d}{d\tau}\hat{D}^{\alpha}\Big)\Big(\frac{d}{d\tau}\hat{D}^{\beta}\Big)\Big|_{\gamma=1}\Big]
$$

(cf. Eqs. (8.4) and (8.33)). Here Z is the missing renormalization constant. This effect is analogous to the effect of the vacuum gravitational waves [23]. The difference is only in the ways in which these effects appear in the formalism and in the dimensions of the coupling constants. The dimension of the coupling constant causes that the gravitational effect does not boil down to a mere renormalization [23].

Removing the limitation that the metric contains no horizon is a separate problem since in this case the total radiation energy is infinite but the reason why it becomes infinite can be pointed out straight away. What becomes wrong is the assumption of asymptotic stationarity of the sources in the future which leads to Eq. (7.35) and hence to Eq. (7.36). Since τ is an external time, as $\tau\to\infty$ the sources moving in the tube hit the event horizon. Therefore, the asymptotic stationarity gets replaced by the condition that at $\tau\to\infty$ the sources remain finite along with their internal derivatives. The growth of the vertex function at i^+ should then become by one power of s faster.

Finally, the assumption that the support of the physical sources is confined to a space-time tube may seem technical and minor but is in fact physical and inadmissibly restrictive. As pointed out in Sec.4, the significance of this assumption is in the fact that it excludes the radiation of charge. In the external-field problem this covers interesting cases but in the self-consistent problem this is merely wrong since what has been calculated above is exactly the radiation of the gravitational charge. Obtaining the backreaction equations is impossible without letting the sources out of the tube.

Acknowledgments

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Appendix A. Identities for the vertex operators. Use of the Jacobi identities.

The functions $A_i^k(\Box_m, \Box_n)$ and $B_i^k(\Box_m, \Box_n)$ in the asymptotic expansions of the form factors (2.33) are given in Ref.[18] as linear combinations with rational coefficients of the vertex operators (2.44)

$$F(k_m, k_n) \equiv \Big(\frac{\partial}{\partial j_m}\Big)^{k_m} \Big(\frac{\partial}{\partial j_n}\Big)^{k_n} \frac{\log(j_m\Box_m / j_n\Box_n)}{j_m\Box_m - j_n\Box_n}\Big|_{j=1} \quad , \quad m \neq n \quad . \tag{A.1}$$

Expression (A.1) can also be written in the form

$$F(k_m, k_n) = \Box_m^{k_m}\Box_n^{k_n} \Big(\frac{\partial}{\partial\Box_m}\Big)^{k_m} \Big(\frac{\partial}{\partial\Box_n}\Big)^{k_n} \frac{\log(\Box_m/\Box_n)}{\Box_m - \Box_n} \quad . \tag{A.2}$$

The results in [18] simplify drastically if one takes into account the constraints that exist between the functions (A.1) with different k_m and k_n . The constraint equation reads

$$F(k_m, k_n + 1) + F(k_m + 1, k_n) = -(k_m + k_n + 1)F(k_m, k_n) \tag{A.3}$$

and is obtained by acting with the operator

$$\Box_m^{k_m}\Box_n^{k_n} \Big(\frac{\partial}{\partial\Box_m}\Big)^{k_m} \Big(\frac{\partial}{\partial\Box_n}\Big)^{k_n} \tag{A.4}$$

on the easily verifiable identity

$$\Big(\Box_m \frac{\partial}{\partial\Box_m} + \Box_n \frac{\partial}{\partial\Box_n}\Big) \frac{\log(\Box_m/\Box_n)}{\Box_m - \Box_n} \equiv -\frac{\log(\Box_m/\Box_n)}{\Box_m - \Box_n} \quad . \tag{A.5}$$

The first step in simplifying the functions $A_i^k(\Box_m, \Box_n)$ and $B_i^k(\Box_m, \Box_n)$ is a removal of the dimensionless coefficients $\Box_m/\Box_n$ that some of these functions have in front of $F(k_m, k_n)$ [18]. This removal can be done so that there remain linear combinations of $F(k_m, k_n)$ with numerical coefficients and rational additions. Indeed, by writing

$$\frac{\Box_m}{\Box_n} F(k_m, k_n) \equiv \frac{\Box_m - \Box_n}{\Box_n} F(k_m, k_n) + F(k_m, k_n) \tag{A.6}$$

and commuting $(\Box_m - \Box_n)$ with the derivatives $\partial/\partial\Box$ in (A.2), one obtains

$$\frac{\Box_m - \Box_n}{\Box_n}F(0,k_n) = (-1)^{k_n}\frac{(k_n-1)!}{\Box_n} + k_n F(0,k_n-1) \quad , \quad k_n \neq 0 \quad , \tag{A.7}$$

$$\begin{aligned}\frac{\Box_m - \Box_n}{\Box_n}F(k_m,k_n) &= -k_n\frac{\Box_m - \Box_n}{\Box_n}F(k_m-1,k_n) + (k_m+k_n)F(k_m,k_n-1) \\ &\quad + k_m(k_m+k_n-1)F(k_m-1,k_n-1) \quad , \quad k_m \neq 0\ ,\ k_n \neq 0 \quad .\end{aligned} \tag{A.8}$$

The latter identity applied k_m times with a subsequent use of (A.7) and (A.3) brings to the following reduction formula:

$$\frac{\Box_m - \Box_n}{\Box_n}F(k_m,k_n) = (k_m+k_n)F(k_m,k_n-1) + (-1)^{k_m+k_n}\frac{k_m!(k_n-1)!}{\Box_n} \quad , \quad k_m \neq 0\ ,\ k_n \neq 0 \quad . \tag{A.9}$$

This formula is useful also in the case where there is a dimensional coefficient $1/\Box_n$ in front of $F(k_m,k_n)$ [18] since it enables one to convert the $1/\Box_n$ into the $1/\Box_m$ and vice versa. For that, it suffices to multiply Eqs. (A.7) and (A.9) by $1/\Box_m$.

The relations above make it possible to bring the asymptotic form factors in [18] to their final form in (2.39)-(2.45). The strategy of this calculation is as follows. First, the relations (A.6)-(A.9) are used to get rid of the coefficients $\Box_m/\Box_n$ in front of $F(k_m,k_n)$ and, if needed, to replace the coefficients $1/\Box_n$ with $1/\Box_m$. Next, the relation (A.3) is used in the thus obtained linear combinations of $F(k_m,k_n)$ to reduce the difference between k_m and k_n . For example,

$$F(0,4) + F(4,0) = 2F(2,2) - 48F(1,1) + 24F(0,0) \quad . \tag{A.10}$$

The equality of the exponents k_m and k_n ensures ultimately the positive definiteness of the total radiation energy.

For illustration, consider two specific examples of the functions ${B_i}^k$ in [18]:

$${B_{12}}^1(\Box_2,\Box_3) = -\frac{2}{3\Box_2\Box_3} - \frac{1}{3\Box_2}F(1,0) - \frac{1}{3\Box_3}F(0,1) \tag{A.11}$$

and

$$\begin{aligned}{B_7}^1(\Box_2,\Box_3) &= \frac{1}{24}F(1,0) + \frac{1}{24}F(0,1) + \frac{1}{96}F(2,0) + \frac{1}{96}F(0,2) + \frac{19}{48}F(1,1) \\ &\quad - \frac{1}{24}F(3,1) - \frac{1}{24}F(1,3) + \frac{\Box_3}{\Box_2}\Big(\frac{1}{96}F(2,0) - \frac{1}{96}F(1,1)\Big) + \frac{\Box_2}{\Box_3}\Big(-\frac{1}{96}F(1,1) + \frac{1}{96}F(0,0)\Big) \quad .\end{aligned} \tag{A.12}$$

In (A.11), make the factors $1/\Box$ in front of the F 's uniform, and use (A.3). As a result, ${B_{12}}^1$ proves to be a tree operator:

$$ {B_{12}}^1(\Box_2, \Box_3) = -\frac{1}{3\Box_2\Box_3} \quad . \tag{A.13} $$

In (A.12), first remove the factors $\Box_3/\Box_2$, $\Box_2/\Box_3$ with the aid of (A.7) and (A.9), and next use (A.3). The result is

$$ {B_7}^1(\Box_2, \Box_3) = \frac{1}{12}F(2,2) - \frac{1}{6}F(1,1) \quad . \tag{A.14} $$

Apart from the simplification of the form factors, the expression for the effective action in [18] needs to be improved in the terms with the commutator curvature. In the work [15-18], the basis of nonlocal gravitational invariants was expressed from the outset through the source of the metric curvature by the use of the Bianchi identities. It is an omission of this work that a similar procedure has not been applied to the commutator curvature. By differentiating and contracting the Jacobi identities (1.5) one obtains the equation

$$ \Box\hat{\mathcal{R}}_{\alpha\beta} = \nabla^\gamma\nabla_\beta\hat{\mathcal{R}}_{\alpha\gamma} - \nabla^\gamma\nabla_\alpha\hat{\mathcal{R}}_{\beta\gamma} \tag{A.15} $$

which can be solved iteratively to express the commutator curvature $\hat{\mathcal{R}}_{\alpha\beta}$ through its source $\hat{J}_\alpha = \nabla^\gamma\hat{\mathcal{R}}_{\alpha\gamma}$ and the initial data. The iteration procedure is started by commuting the covariant derivatives on the right-hand side of (A.15) with the aid of Eq. (1.10). Neglecting the commutators, one obtains to lowest order in $\Re$

$$ \Box\hat{\mathcal{R}}_{\alpha\beta} = \nabla_\beta\hat{J}_\alpha - \nabla_\alpha\hat{J}_\beta + O[\Re^2] \quad . \tag{A.16} $$

The solution of this equation with the retarded Green function:

$$ \hat{\mathcal{R}}_{\alpha\beta} = \frac{1}{\Box}\Big(\nabla_\beta\hat{J}_\alpha - \nabla_\alpha\hat{J}_\beta\Big) + O[\Re^2] \tag{A.17} $$

is the solution with no incoming wave of the vector connection field. (See Ref.[23] for the proof of a similar assertion in the case of the gravitational field.)

From the results in [18] (with the vertex operators treated as above), the contribution of the commutator curvature to the scalar I_2, Eq. (2.39), obtains originally in the form

$$ \frac{1}{2}f(\Box_1, \Box_2)\Big\{\Big(\frac{1}{\Box_1} + \frac{1}{\Box_2}\Big)\nabla_\alpha\hat{\mathcal{R}}_1^{\alpha\lambda}\nabla_\beta\hat{\mathcal{R}}_2{}^\beta{}_\lambda + \hat{\mathcal{R}}_1{}^{\alpha\beta}\hat{\mathcal{R}}_{2\,\alpha\beta}\Big\} \tag{A.18} $$

where

$$f(\Box_1, \Box_2) = \frac{1}{6}F(2,2) - \frac{1}{3}F(1,1) \quad . \tag{A.19}$$

The form (A.18) is redundant because the commutator curvature is constrained by the Jacobi identities. By using (A.17), one finds

$$\hat{\mathcal{R}}_1^{\alpha\beta}\hat{\mathcal{R}}_{2\,\alpha\beta} = 2\frac{1}{\Box_1\Box_2}\nabla_\beta \hat{J}_1^{\alpha}\nabla^\beta \hat{J}_{2\,\alpha} - 2\frac{1}{\Box_1\Box_2}\nabla_\beta \hat{J}_1^{\alpha}\nabla_\alpha \hat{J}_2^{\beta} \tag{A.20}$$

where in the first term use can be made of the relation [15]

$$2\nabla_1\nabla_2 = (\nabla_1 + \nabla_2)^2 - \Box_1 - \Box_2 \quad . \tag{A.21}$$

The contribution of the operator $(\nabla_1 + \nabla_2)^2$ to (A.20) is of the form

$$\frac{1}{\Box_1\Box_2}(\nabla_1 + \nabla_2)^2 \hat{J}_1^{\alpha}\hat{J}_{2\,\alpha} \equiv \Box\Big[\Big(\frac{1}{\Box}\hat{J}^{\alpha}\Big)\Big(\frac{1}{\Box}\hat{J}_{\alpha}\Big)\Big] \quad . \tag{A.22}$$

Since here appears an overall $\Box$ operator acting at the observation point, the respective contribution to (A.18) can be discarded (see Sec.2). As a result, expression (A.18) takes the form

$$-\frac{f(\Box_1, \Box_2)}{\Box_1\Box_2}\nabla_\beta \hat{J}_1^{\alpha}\nabla_\alpha \hat{J}_2^{\beta} \quad . \tag{A.23}$$

Finally, after the scalar I_2 in (2.39) has been expressed as a quadratic combination of the conserved currents, the derivatives acting on the individual currents can all be made overall owing to the conservation laws (1.9). Thus, up to $O[\Re^3]$

$$\Big(\nabla_\beta \hat{J}_1^{\alpha}\Big)\Big(\nabla_\alpha \hat{J}_2^{\beta}\Big) = \nabla_\beta\nabla_\alpha\Big(\hat{J}_1^{\alpha}\hat{J}_2^{\beta}\Big) \quad , \tag{A.24}$$

$$\Big(\nabla_\alpha\nabla_\beta J_1^{\mu\nu}\Big)\Big(\nabla_\mu\nabla_\nu J_2^{\alpha\beta}\Big) = \nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu\Big(J_1^{\mu\nu}J_2^{\alpha\beta}\Big) \quad . \tag{A.25}$$

In this way the final result (2.39)-(2.45) is obtained.

Appendix B. Alternative approach to the non-scalar vertices.

There is an alternative to the special consideration of the non-scalar vertices in Secs.4 and 7. It is based on the fact that up to terms $O[\Re^3]$ the overall derivatives in (2.39) can be commuted with the operator $\log(-\Box)$. Retaining only the vertex terms we have from (2.36) and (2.39)

$$
\begin{aligned}
T^{\mu\nu}_{\text{vac}}\Big|_{\mathcal{I}^+} &= \frac{1}{(4\pi)^2}\,\mathrm{tr}\,\Big\{\nabla^\mu\nabla^\nu\log(-\Box)\hat{V}_{\text{scalar}} \\
&\quad + \nabla^\mu\nabla^\nu\nabla_\alpha\nabla_\beta\log(-\Box)\Big(\hat{V}^{\alpha\beta}_{\text{cross}} + \hat{V}^{\alpha\beta}_{\text{vect}}\Big) \\
&\quad + \nabla^\mu\nabla^\nu\nabla_\alpha\nabla_\beta\nabla_\gamma\nabla_\sigma\log(-\Box)\hat{V}^{\alpha\beta\gamma\sigma}_{\text{grav}} + \text{trees}\Big\} \quad . \qquad \text{(B.1)}
\end{aligned}
$$

The derivatives act now at a point at $\mathcal{I}^+$ and, moreover, we need only the senior terms of expansions like (4.59) and (4.76):

$$
\nabla^\mu\nabla^\nu \;\to\; \nabla^\mu u\nabla^\nu u\frac{\partial^2}{\partial u^2} \quad , \quad \nabla^\mu\nabla^\nu\nabla_\alpha\nabla_\beta \;\to\; \nabla^\mu u\nabla^\nu u\nabla_\alpha u\nabla_\beta u\frac{\partial^4}{\partial u^4} \quad , \quad \text{etc.} \qquad \text{(B.2)}
$$

Thus, making use of Eqs. (1.36) and (4.18), we obtain

$$
\begin{aligned}
M(-\infty) - M(\infty) &= -\lim_{u\to\infty}\frac{2}{(4\pi)^2}\int d^2\mathcal{S}(\phi)\;\mathrm{tr}\,\Big\{\frac{\partial^2}{\partial u^2}D_{\mathbf{1}}(u,\phi|\hat{V}_{\text{scalar}}) \\
&\quad + \frac{\partial^4}{\partial u^4}D_{\mathbf{1}}(u,\phi|\hat{V}^{\alpha\beta}_{\text{cross}})\nabla_\alpha u\nabla_\beta u + \frac{\partial^4}{\partial u^4}D_{\mathbf{1}}(u,\phi|\hat{V}^{\alpha\beta}_{\text{vect}})\nabla_\alpha u\nabla_\beta u \\
&\quad + \frac{\partial^6}{\partial u^6}D_{\mathbf{1}}(u,\phi|\hat{V}^{\alpha\beta\gamma\sigma}_{\text{grav}})\nabla_\alpha u\nabla_\beta u\nabla_\gamma u\nabla_\sigma u\Big\} \quad . \qquad \text{(B.3)}
\end{aligned}
$$

The price for getting rid of the derivatives is that now we have to deal with the moments $D_{\mathbf{1}}$ of the non-scalar vertices. Since these vertices have different powers of growth at i^+, Eq. (6.41) is to be applied in each case separately. Let us introduce a notation for the tensors at i^+ that figure in the asymptotic expressions for the vertices, Eqs. (7.40)-(7.42):

$$
\Delta^{\alpha\beta}_{\text{cross}}(x)\Big|_{i^+} = \frac{1}{48}\int_{-\infty}^{\infty} d\tau\Big(\frac{d^2}{d\tau^2}D^{\alpha\beta}\Big)\Big(\frac{d^2}{d\tau^2}\hat{D}^Q\Big) \quad , \qquad \text{(B.4)}
$$

$$
\Delta^{\alpha\beta}_{\text{vect}}(x)\Big|_{i^+} = \frac{1}{48(1-\gamma^2)}\int_{-\infty}^{\infty} d\tau\Big(\frac{d}{d\tau}\hat{D}^\alpha\Big)\Big(\frac{d}{d\tau}\hat{D}^\beta\Big) \quad , \qquad \text{(B.5)}
$$

$$\Delta^{\alpha\beta\gamma\sigma}_{\rm grav}(x)\Big|_{i^+} \;=\; -\frac{\hat{1}}{180\times 32}\int\limits_{-\infty}^{\infty} d\tau\Big(\frac{d^2}{d\tau^2}D^{\alpha\beta}\Big)\Big(\frac{d^2}{d\tau^2}D^{\gamma\sigma}\Big)\quad . \tag{B.6}$$

With the behaviours of the V 's in (7.40)-(7.42), the algorithm (6.41) yields

$$\begin{aligned} D_{\mathbf{1}}(u,\phi|V^{\alpha\beta})\Big|_{u\to\infty} = \frac{u^4}{4\pi}\int\limits_0^1 d\gamma\,\gamma^2(1-\gamma^2)^3\int d^2\mathcal{S}(\bar\phi)\,(1-\gamma n\bar n)^{-5} \\ \times\Big(g^{\alpha}{}_{\bar\alpha}(x,\bar x)g^{\beta}{}_{\bar\beta}(x,\bar x)\Delta^{\bar\alpha\bar\beta}(\bar x)\Big)\Big|_{\substack{x\,\to\,\mathcal{I}^+[u,\phi]\\ \bar x\,\to\, i^+[\gamma,\bar\phi]}} \end{aligned} \tag{B.7}$$

for both $\hat{V}^{\alpha\beta}_{\rm cross}$ and $\hat{V}^{\alpha\beta}_{\rm vect}$, and

$$\begin{aligned} D_{\mathbf{1}}(u,\phi|V^{\alpha\beta\gamma\sigma})\Big|_{u\to\infty} = \frac{u^6}{4\pi}\int\limits_0^1 d\gamma\,\gamma^2(1-\gamma^2)^4\int d^2\mathcal{S}(\bar\phi)\,(1-\gamma n\bar n)^{-7} \\ \times\Big(g^{\alpha}{}_{\bar\alpha}(x,\bar x)g^{\beta}{}_{\bar\beta}(x,\bar x)g^{\gamma}{}_{\bar\gamma}(x,\bar x)g^{\sigma}{}_{\bar\sigma}(x,\bar x)\Delta^{\bar\alpha\bar\beta\bar\gamma\bar\sigma}(\bar x)\Big)\Big|_{\substack{x\,\to\,\mathcal{I}^+[u,\phi]\\ \bar x\,\to\, i^+[\gamma,\bar\phi]}} \end{aligned} \tag{B.8}$$

for $\hat{V}^{\alpha\beta\gamma\sigma}_{\rm grav}$. Here the appearance of

$$n\bar n = n_i(\phi)n^i(\bar\phi) \tag{B.9}$$

is owing to Eq. (4.16), and we didn't forget *the propagators of parallel transport* connecting the point at $\mathcal{I}^+$ with the point at i^+ .

The powers of u in expressions (B.7) and (B.8) are just the ones needed for the quantity (B.3) to be finite and nonvanishing but the weight of calculation transfers now to carrying out the integration over the directions of radiation i.e. over the angles ϕ at $\mathcal{I}^+$. As seen from (B.3) and (B.7)-(B.8), one is to do the integrals

$$\begin{aligned} \Pi_{\bar\alpha\bar\beta}(\bar x)\Big|_{i^+} &= \int d^2\mathcal{S}(\phi)\,(1-\gamma n\bar n)^{-5} \\ &\times\;\Big(\nabla_\alpha u(x)\nabla_\beta u(x)g^{\alpha}{}_{\bar\alpha}(x,\bar x)g^{\beta}{}_{\bar\beta}(x,\bar x)\Big)\Big|_{\substack{x\,\to\,\mathcal{I}^+[u,\phi]\\ \bar x\,\to\, i^+[\gamma,\bar\phi]}}\quad , \end{aligned} \tag{B.10}$$

$$\begin{aligned} \Pi_{\bar\alpha\bar\beta\bar\gamma\bar\sigma}(\bar x)\Big|_{i^+} &= \int d^2\mathcal{S}(\phi)\,(1-\gamma n\bar n)^{-7} \\ &\times\;\Big(\nabla_\alpha u(x)\nabla_\beta u(x)\nabla_\gamma u(x)\nabla_\sigma u(x)g^{\alpha}{}_{\bar\alpha}(x,\bar x)g^{\beta}{}_{\bar\beta}(x,\bar x)g^{\gamma}{}_{\bar\gamma}(x,\bar x)g^{\sigma}{}_{\bar\sigma}(x,\bar x)\Big)\Big|_{\substack{x\,\to\,\mathcal{I}^+[u,\phi]\\ \bar x\,\to\, i^+[\gamma,\bar\phi]}}. \end{aligned} \tag{B.11}$$

Since both points of the propagators $g^{\alpha}{}_{\bar{\alpha}}(x,\bar{x})$ are at the future infinity, one may use the flat-spacetime expressions for these propagators. The integration is conveniently carried out in the Minkowski frame (1.25) for both points x and $\bar{x}$. In this frame,

$$g^{\alpha}{}_{\bar{\alpha}}(x,\bar{x}) = \delta^{\alpha}{}_{\bar{\alpha}} \quad , \quad \nabla_{\alpha} u(x) = \delta^{0}_{\alpha} - \delta^{i}_{\alpha} n_i \quad , \tag{B.12}$$

and there emerge only integrals of the form

$$\int d^2\mathcal{S}(\phi)\, (1-\gamma n\bar{n})^{-p} n_i \cdots n_j \tag{B.13}$$

summarized in the table below.

When contracting the Π 's obtained in the Minkowski frame with the Δ 's in Eqs. (B.4)-(B.6), it should be taken into account that the Minkowski components of any of the Δ 's can be expressed as follows:

$$\begin{aligned} \Delta^{0\cdots} = \gamma \nabla_{\alpha} r \Delta^{\alpha\cdots} \quad , \quad n_i \Delta^{i\cdots} = \nabla_{\alpha} r \Delta^{\alpha\cdots} \quad , \\ \delta_{ik} \Delta^{ik\cdots} = (g_{\alpha\beta} + \gamma^2 \nabla_{\alpha} r \nabla_{\beta} r) \Delta^{\alpha\beta\cdots} \end{aligned} \tag{B.14}$$

by virtue of the conservation equations (7.23),(7.24).

In this way after many cancellations we obtain

$$\Pi_{\alpha\beta} \Delta^{\alpha\beta}(x)\Big|_{i^+} = \frac{4\pi}{3} \frac{1}{(1-\gamma^2)^3} g_{\alpha\beta} \Delta^{\alpha\beta}(x)\Big|_{i^+} \tag{B.15}$$

for both $\Delta^{\alpha\beta}_{\rm cross}$ and $\Delta^{\alpha\beta}_{\rm vect}$, and

$$\Pi_{\alpha\beta\gamma\sigma} \Delta^{\alpha\beta\gamma\sigma}(x)\Big|_{i^+} = \frac{4\pi}{5} \frac{1}{(1-\gamma^2)^4} g_{\alpha\beta} g_{\gamma\sigma} \Delta^{(\alpha\beta\gamma\sigma)}(x)\Big|_{i^+} \tag{B.16}$$

for $\Delta^{\alpha\beta\gamma\sigma}_{\rm grav}$ where the latter tensor appears symmetrized. The use of these contractions in (B.7),(B.8) and (B.3) yields the same final result as in the main text, Eq. (8.4).

The integrals (B.13) are elementary but it is useful to have them. Therefore, we present the ones needed for the calculation above. Denoting

$$n\gamma = n_i(\phi)\gamma^i \quad , \quad \gamma_i = \delta_{ik}\gamma^k \quad , \quad \gamma^2 = \delta_{ik}\gamma^i\gamma^k$$

we have

$$\frac{1}{4\pi} \int d^2\mathcal{S}(\phi)\, \frac{1}{(1-n\gamma)^3} \quad = \quad \frac{1}{(1-\gamma^2)^2} \quad ,$$

$$\begin{aligned}
\frac{1}{4\pi}\int d^2\mathcal{S}(\phi)\,\frac{1}{(1-n\gamma)^4} &= \frac{1}{3}\frac{(3+\gamma^2)}{(1-\gamma^2)^3} \quad ,\\
\frac{1}{4\pi}\int d^2\mathcal{S}(\phi)\,\frac{1}{(1-n\gamma)^5} &= \frac{(1+\gamma^2)}{(1-\gamma^2)^4} \quad ,\\
\frac{1}{4\pi}\int d^2\mathcal{S}(\phi)\,\frac{1}{(1-n\gamma)^6} &= \frac{1}{5}\frac{(5+10\gamma^2+\gamma^4)}{(1-\gamma^2)^5} \quad ,\\
\frac{1}{4\pi}\int d^2\mathcal{S}(\phi)\,\frac{1}{(1-n\gamma)^7} &= \frac{1}{3}\frac{(1+3\gamma^2)(3+\gamma^2)}{(1-\gamma^2)^6} \quad ,\\
\frac{1}{4\pi}\int d^2\mathcal{S}(\phi)\,\frac{n_i}{(1-n\gamma)^5} &= \frac{1}{3}\frac{(5+\gamma^2)}{(1-\gamma^2)^4}\gamma_i \quad ,\\
\frac{1}{4\pi}\int d^2\mathcal{S}(\phi)\,\frac{n_i n_k}{(1-n\gamma)^5} &= \frac{1}{3}\frac{1}{(1-\gamma^2)^3}\Big(\delta_{ik}+6\frac{\gamma_i\gamma_k}{1-\gamma^2}\Big) \quad ,\\
\frac{1}{4\pi}\int d^2\mathcal{S}(\phi)\,\frac{n_i}{(1-n\gamma)^7} &= \frac{1}{15}\frac{(35+42\gamma^2+3\gamma^4)}{(1-\gamma^2)^6}\gamma_i \quad ,\\
\frac{1}{4\pi}\int d^2\mathcal{S}(\phi)\,\frac{n_i n_k}{(1-n\gamma)^7} &= \frac{1}{15}\frac{(5+3\gamma^2)}{(1-\gamma^2)^5}\delta_{ik}+\frac{8}{15}\frac{(7+3\gamma^2)}{(1-\gamma^2)^6}\gamma_i\gamma_k \quad ,\\
\frac{1}{4\pi}\int d^2\mathcal{S}(\phi)\,\frac{n_i n_k n_p}{(1-n\gamma)^7} &= \frac{8}{15}\frac{(9+\gamma^2)}{(1-\gamma^2)^6}\gamma_i\gamma_k\gamma_p+\frac{1}{15}\frac{(7+\gamma^2)}{(1-\gamma^2)^5}(\delta_{ik}\gamma_p+\delta_{ip}\gamma_k+\delta_{kp}\gamma_i) \quad ,\\
\frac{1}{4\pi}\int d^2\mathcal{S}(\phi)\,\frac{n_i n_j n_k n_p}{(1-n\gamma)^7} &= \frac{16}{3}\frac{1}{(1-\gamma^2)^6}\gamma_i\gamma_j\gamma_k\gamma_p+\frac{1}{15}\frac{1}{(1-\gamma^2)^4}(\delta_{ij}\delta_{kp}+\delta_{ik}\delta_{jp}+\delta_{ip}\delta_{jk})\\
&+ \frac{8}{15}\frac{1}{(1-\gamma^2)^5}(\delta_{ij}\gamma_k\gamma_p+\delta_{ik}\gamma_j\gamma_p+\delta_{ip}\gamma_j\gamma_k+\delta_{jk}\gamma_i\gamma_p+\delta_{jp}\gamma_i\gamma_k+\delta_{kp}\gamma_i\gamma_j).
\end{aligned}$$

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Captions to the figures

Fig.1. The Bondi-Sachs frame with the central geodesic $r = 0$ and the light cones of equal retarded time.

Fig.2. Shown are the past light cone of a point x, an arbitrary point o on its surface, and both sheets of the light cone of o. (The line $r = 0$ is the central geodesic.) The past light cone of x lies outside the light cone of o.

Fig.3. The past hyperboloids of x, $\sigma(x, \bar{x}) = q$, inscribed in the past light cone of x, $\sigma(x, \bar{x}) = 0$.

Fig.4. The past light cone of x in Fig.2 after x has moved to $\mathcal{I}^+$ along the radial geodesic shown bold. The resultant null hyperplane has the parameters u, ϕ of this geodesic.

Fig.5. Shown are the support tube of a physical source J, the compact support of $\dot{J}$, and the earliest hyperboloid of x that still crosses the latter support. The broken lines bound the causal future of the support of $\dot{J}$.

Fig.6. Penrose diagram for the section of fixed angles in the Bondi-Sachs coordinates. The exterior of the light cone of o is divided into four subdomains bounded by two future light cones $u = u_1$ and $u = u_2$, and the surface of the tube $r < r_0$.

Fig.7. Lorentzian section Γ of a spherically symmetric spacetime. The broken line is the mapping on Γ of a timelike radial geodesic. The lines $T^{\pm} = \tau$ bound the mapping on Γ of a spacelike hyperplane. The bold line is the world line of the support shell (or the boundary of the support tube) of the source.

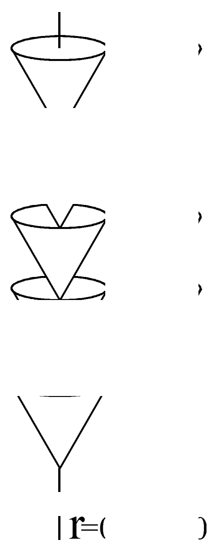

Fig.1

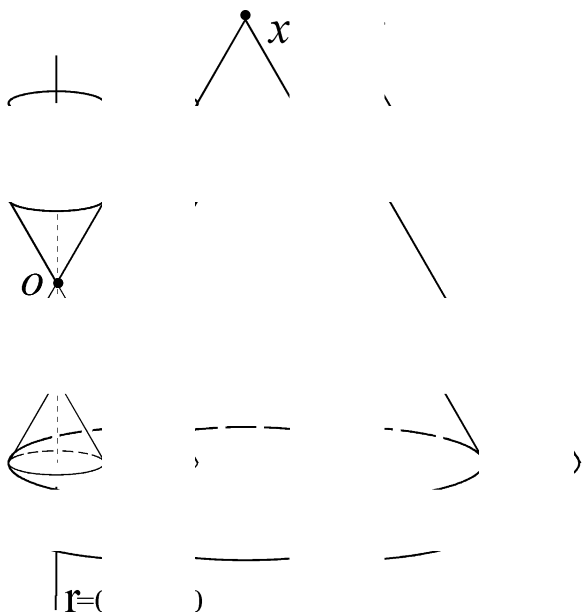

Fig.2

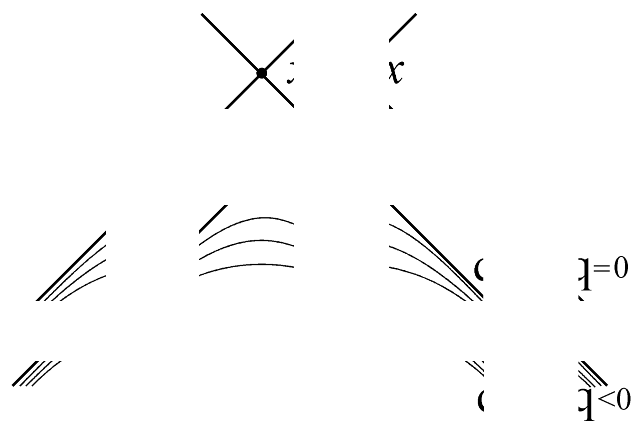

Fig.3

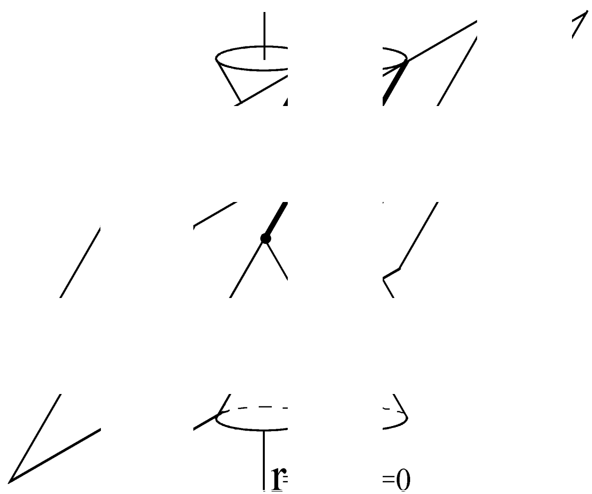

Fig.4

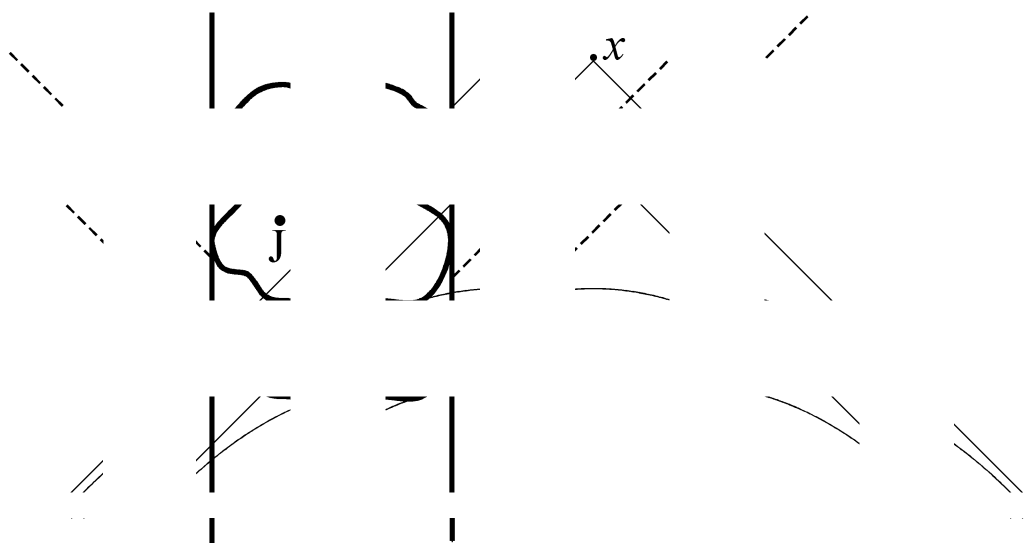

Fig.5

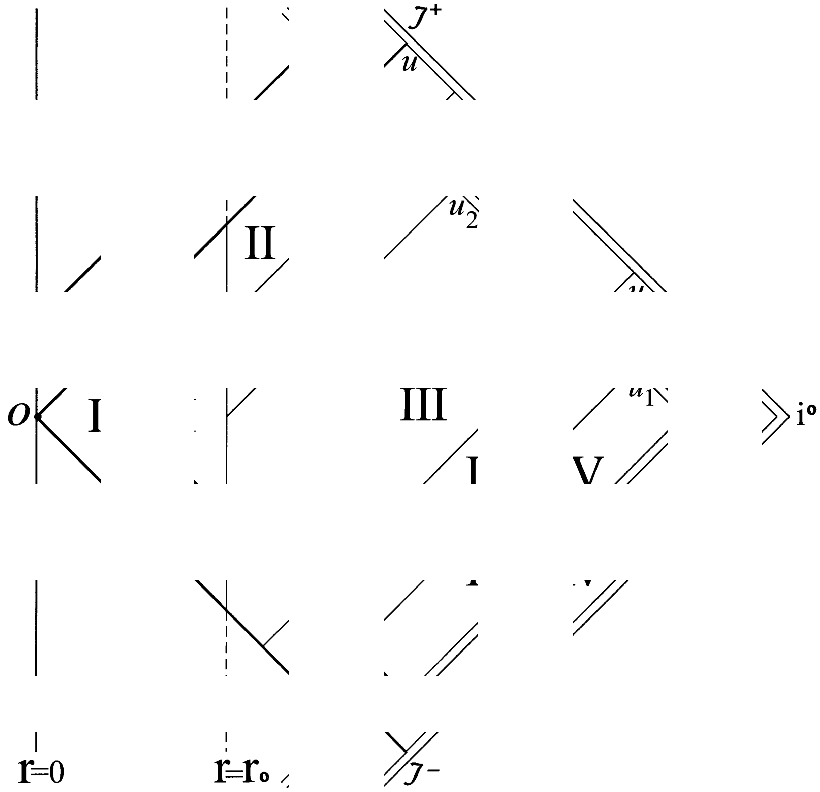

Fig.6

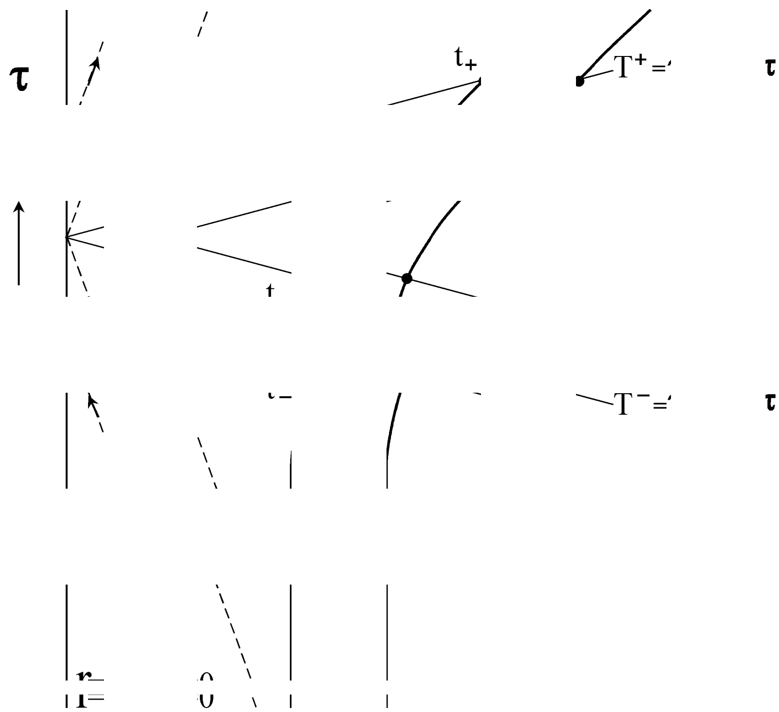

Fig.7